Quantum Vortices and Vortex Lattices

Lecture 3: The Ginzburg-Landau Equations

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Zagreb, May 2026

Ginzburg-Landau Equations

Equilibrium states of superconductors (macroscopically) and of the $U(1)$ Higgs model of particle physics are described by the Ginzburg-Landau equations (GLE):

$$\begin{array}{c} -\Delta_A\Psi = \kappa^2(1-|\Psi|^2)\Psi \\ \operatorname{curl}^2 A = \operatorname{Im}(\bar{\Psi}\nabla_A\Psi) \end{array} \qquad (1)$$

where $(\Psi, A) : \mathbb{R}^d \to \mathbb{C} \times \mathbb{R}^d$, $d = 2, 3$, $\nabla_A = \nabla - iA$, $\Delta_A = \nabla_A^2$, the covariant derivative and covariant Laplacian, respectively, and κ is the Ginzburg-Landau (GL) material constant. BC: $|\Psi| \to 1$ as $|x| \to \infty$.

The GLE are the Euler-Lagrange equations for the GL energy functl

$$E_Q(\Psi, A) = \frac{1}{2}\int_Q \left\{|\nabla_A\Psi|^2 + |\operatorname{curl} A|^2 + \frac{\kappa^2}{2}(|\Psi|^2-1)^2\right\}, \qquad (2)$$

for $\int_Q \operatorname{curl} A$ (average magnetic field) fixed, where $Q \subset \mathbb{R}^2$.

Meaning of Ψ and A

Superconductivity. $\Psi : \mathbb{R}^d \to \mathbb{C}$ is called the *order parameter*;
$|\Psi|^2$ is the density of (Cooper pairs of) superconducting electrons;
$A : \mathbb{R}^d \to \mathbb{R}^d$ is the magnetic potential;
$\mathrm{Im}(\bar{\Psi}\nabla_A \Psi)$ is the superconducting current.

Particle physics. Ψ and A are the Higgs and $U(1)$ gauge (electro-magnetic) fields, respectively. (Part of Weinberg - Salam model of electro-weak interactions/ a standard model.)

Geometrically, A is a connection on the principal $U(1)$- bundle $\mathbb{R}^d \times U(1)$, and Ψ, a section of the associated line bundle.

Similar equations appear in other areas of physics and material sciences.

Extensions: Yang-Mills-Higgs and Seiberg-Witten equations

Main objective: Describe ground states and symmetry breaking.

Ground States and Symmetries

Ground states ($\equiv$ vacua in particle physics) are solutions with smallest energy (per unit volume)

Expect: Ground states have maximal possible symmetry.

GLE symmetries (i.e. transforms mapping solutions to solutions):

Gauge symmetry: for any C^1 function $\gamma : \mathbb{R}^d \to \mathbb{R}$,

$$T^{\mathrm{gauge}}_{\chi} : (\Psi(x), A(x)) \mapsto (e^{i\chi(x)}\Psi(x), A(x) + \nabla\chi(x)); \qquad (3)$$

Translational/rotational symmetry: $\forall h \in \mathbb{R}^d / \forall \rho \in SO(d)$,

$$T^{\mathrm{transl}}_{h} : (\Psi(x), A(x)) \mapsto (\Psi(x+h), A(x+h))/ \ \ldots . \qquad (4)$$

Expect ground states to be gauge-translationally invariant:

$$\exists \chi_h(x) : \ \ T^{\mathrm{transl}}_{h}(\Psi, A) = T^{\mathrm{gauge}}_{\chi_h}(\Psi, A), \ \forall h \in \mathbb{R}^2. \qquad (5)$$

((5) implies that $\chi_h(x)$ is a co-cycle and $\chi_h(x) := \frac{1}{2} b \cdot (x \wedge h)$.)

As common, we will consider $\mathbb{R}^2$ (the cylindrical geometry).

Symmetry Breaking

The gauge-translationally invariant solutions: $u_b := (\Psi = 0, A = A^b)$, where $\mathrm{curl}\, A^b = b$=const.

We label solns by the average magnetic field $\lim_{\Omega \to \mathbb{R}^2} \frac{1}{|\Omega|} \int_\Omega \mathrm{curl}\, A = b$.

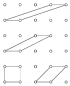
Stability (linearized): Sol u_* stable $\Longleftrightarrow$ Hessian $E(u_*) \geq 0$

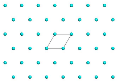
Theorem (Emergence of inhomogeneous solutions)

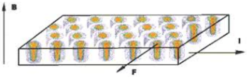
(i) The gauge-translationally invariant ground state is stable for $b > \kappa^2$ *and unstable for* $b < \kappa^2$*;*

(ii) At $b = \kappa^2$ *a new stable solution emerges, which is gauge - invariant under lattice translations;*

(iii) The lattice with the lowest energy per unit area is the hexagonal one.







Lattice-gauge invariance and quantization of magnetic flux

Emerging solutions (Ψ, A) have reduced symmetry: from $\mathbb{R}^2-$ gauge translations to lattice-gauge ones

$$T_s^{\mathrm{transl}}(\Psi, A) = T_{g_s}^{\mathrm{gauge}}(\Psi, A), \ \forall s \in \lambda, \tag{6}$$

where λ is a lattice in $\mathbb{R}^2 \simeq \mathbb{C}$. Let ω be a fundamental cell of λ.

Theorem. If (Ψ, A) satisfies (6), then $\forall s, t \in \lambda, x \in \mathbb{R}$, $c(g) := \frac{1}{2\pi}(g_t(x+s) + g_s(x) - g_s(x+t) - g_t(x))$ is an integer &

$$\frac{1}{2\pi}\int_\omega \operatorname{curl} A = c(\chi) \in \mathbb{Z}.$$

By comparison, finite energy states (Ψ, A) are classified by the topological degree

$$\deg(\Psi) := \deg\left(\Psi/|\Psi|\Big|_{|x|=R}\right),$$

where $R \gg 1$, and have the total magnetic flux quantized:

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} \operatorname{curl} A = \deg(\Psi) \in \mathbb{Z}.$$

Stability of gauge translationally invariant solution 1

Let $u_b := (\Psi = 0, A = A^b)$, where curl $A^b = b$=const, the gauge transl. invariant solution.

Defition: u_b is stable $\iff E''(u_b) \geq 0$.

$$\begin{aligned} \text{Compute:} \qquad & E''(u_b) = \mathrm{diag}(-\Delta_{A^b} - \kappa^2, \mathrm{curl}^* \, \mathrm{curl}) \\ \implies \qquad & E''(u_b) \geq 0 \iff -\Delta_{A^b} - \kappa^2 \geq 0. \qquad (7) \end{aligned}$$

Proposition. We have

$$b = \inf \sigma(-\Delta_{A^b}).$$

Proof. Introduce the complex covariant derivative

$$\bar{\partial}_{A^b} := \tfrac{1}{2}((\nabla_{A^b})_1 + i(\nabla_{A^b})_2).$$

By the Weitzenböck/(harmonic oscillator) relation,

$$-\Delta_{A^b} - b = (\bar{\partial}_{A^b})^* \bar{\partial}_{A^b} \geq 0.$$

Hence, $-\Delta_{A^b} \geq b$ and

$$b = \inf \sigma(-\Delta_{A^b}) \iff \exists \phi \neq 0 : \bar{\partial}_{A^b} \phi = 0.$$

Now, we solve the eqn $\bar{\partial}_{A^b}\phi = 0$. For this, we find a function f s.t.

$$f^{-1}\bar{\partial}_{A^b}f = \bar{\partial}.$$

We fix the gauge as $A^b(x) = \frac{b}{2}Jx = \frac{b}{2}x^{\perp}$, where $x^{\perp} = (-x_2, x_1)$. Then, we have $A^b_c = i\frac{b}{4}z$. Then solving $f^{-1}\bar{\partial}_{A^b}f = \bar{\partial}$ gives $f = e^{\frac{b}{4}|z|^2}g(z)$, where g is any holomorphic function. We choose $g = 1$, so so that $f = e^{\frac{b}{4}|z|^2}$ which implies

$$e^{-\frac{b}{4}|z|^2}\bar{\partial}_{A^b}e^{\frac{b}{4}|z|^2} = \bar{\partial}.$$

This immediately proves that $\phi \in \operatorname{Null}\bar{\partial}_{A^b}$ if and only if

$$\xi(z) = e^{\frac{b}{4}|z|^2}\phi(z)$$

satisfies $\bar{\partial}\xi = 0$, i.e. $\xi(z)$ is a holomorphic function.
This shows that the eqn $\bar{\partial}_{A^b}\phi = 0$, and therefore the eqn $(-\Delta_{A^b} - b)\phi = 0$, has solns of the form $\phi(z) = e^{-\frac{b}{4}|z|^2}\xi(z)$, where $z = x_1 + ix_2$ and ξ is any holomorphic function. $\square$

Stability of gauge translationally invariant solution 2

We have shown that

$$b = \inf \sigma(-\Delta_{A^b}). \tag{8}$$

This implies that

$$\inf \sigma(E''(u_b)) = \inf \sigma(-\Delta_{A^b} - \kappa^2) = b - \kappa^2 \tag{9}$$

Recall that u_b is stable $\Longleftrightarrow$ $E''(u_b) \geq 0$

$\Longrightarrow$ u_b is stable for $b > \kappa^2$ and stability is lost at $b = \kappa^2$

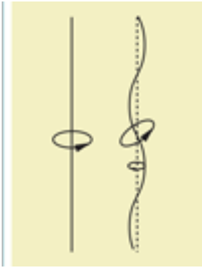
$\Longrightarrow$ bifurcation of new stable ground states of the GLEs. $\square$

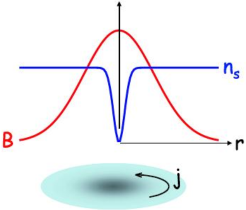
Shape of Lattice Solutions?

Theorem. For any lattice λ with $|\omega|$ large and any integer n, $\exists$ an Abrikosov lattice solution and, near each vertex $s \in \lambda$, it looks like the n-vortex centred at s:

$$\Psi^{(n)}(x-s) = f^{(n)}(r)e^{in\theta} \quad \text{and} \quad A^{(n)}(x-s) = a^{(n)}(r)\nabla(n\theta),$$

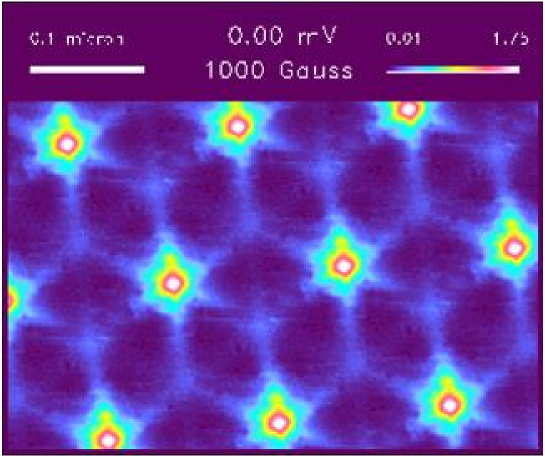
where (r, θ) = polar coordinates of $x - s \in \mathbb{R}^2$.





$(\Psi^{(n)}, A^{(n)})$ is $O(2)-$equivariant solution of degree n:
the *magnetic n-vortex* (superconductors) (excitat. of vortex lattice)
or *Nielsen-Olesen* or *Nambu string* (the particle physics).

Abrikosov Vortex Lattice. Experiment



Time-Dependent Eqns. $U(1)$ Higgs Model

The time-dependent $U(1)$ Higgs model is described by $U(1)-$Higgs (or Maxwell-Higgs) equations

$$
\begin{aligned}
(\partial_t + i\Phi)^2\Psi &= \Delta_A\Psi + \kappa^2(1-|\Psi|^2)\Psi \\
\partial_t(\partial_t A + \nabla\Phi) &= -\operatorname{curl}^2 A + \operatorname{Im}(\bar{\Psi}\nabla_A\Psi),
\end{aligned}
$$

which are coupled (covariant) wave equations.

Time-Dependent Eqns. Superconductivity

In the leading approximation the evolution of a superconductor is described by the gradient-flow-type equations

$$\gamma(\partial_t + i\Phi)\Psi = \Delta_A\Psi + \kappa^2(1 - |\Psi|^2)\Psi$$
$$\sigma(\partial_t A - \nabla\Phi) = -\operatorname{curl}^2 A + Im(\bar{\Psi}\nabla_A\Psi),$$

$Re\gamma \geq 0$, the *time-dependent Ginzburg-Landau equations* or the *Gorkov-Eliashberg-Schmidt equations*. (Earlier versions: Bardeen and Stephen and Anderson, Luttinger and Werthamer.)

The last equation comes from two Maxwell equations, with $-\partial_t E$ neglected, (Ampère's and Gauss' laws) and the relations $J = J_s + J_n$, where $J_s = \mathrm{Im}(\overline{\Psi}\nabla_A\Psi)$, and $J_n = \sigma E$.

Problems (a) WP, (b) stability of vortex lattices and magn. vortices

Stability Definition

Let u_* be a static solution to GLE (Abrikosov vortex lattice, or magnetic vortex). Define the manifold of static solutions

$$\mathcal{M} = \{T_g u_* : g \in G\},$$

where G is (a subgroup of) the symmetry group of GLE,

$$G_{\mathrm{sym}} = H^2(\mathbb{R}^2;\mathbb{R}) \rtimes \mathbb{R}^2 \rtimes SO(2),$$

with $H^2(\mathbb{R}^2;\mathbb{R})$ being the group of the *gauge transformations*, and T_g is the action of G_{sym} on $u = (\Psi, A)$.

Stability/instability: u_* is said to be asympt. stable (unstable), if starting close to $\mathcal{M}$ (in a specified metric (H^1)), solution u_t approaches (departs from) $\mathcal{M}$.

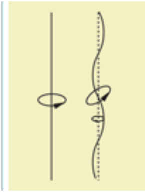
Stability/Instability of Vortices

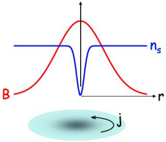
Recall, the *magnetic n-vortices* are “radially symmetric” (more precisely, *equivariant*) solutions of degree n,

$$\Psi^{(n)}(x) = f^{(n)}(r)e^{in\theta} \quad \text{and} \quad A^{(n)}(x) = a^{(n)}(r)\nabla(n\theta),$$

where $n =$ integer and $(r, \theta) =$ polar coordinates of $x \in \mathbb{R}^2$.

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} \operatorname{curl} A^{(n)} = \deg(\Psi^{(n)}) = n \in \mathbb{Z}.$$





Theorem

1. *For $\kappa < 1/2$ (Type I superconductors), all vortices are stable.*
2. *For $\kappa > 1/\sqrt{2}$ (Type II superconductors), the ± 1-vortices are stable, while the n-vortices with $|n| \geq 2$, are not.*

Ginzburg-Landau equations on surfaces

As a model describing superconducting thin membranes or quantum engine, consider the GLE on Riemann surfaces
(= two-dimensional real, orientable and metrizable surfaces).

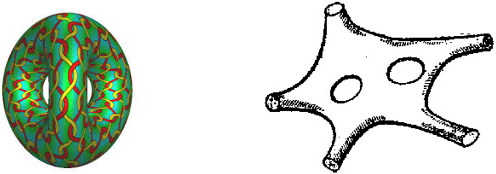


Figure: Compact and non-compact Riemann surfaces.

If the magnetic field $\neq 0$, then the GLE is defined on line bundles over Riemann surfaces:

$$\Delta_A \Psi = \kappa^2(|\Psi|^2 - 1)\Psi, \tag{10a}$$

$$d^* dA = \mathrm{Im}(\bar{\Psi}\nabla_A \Psi). \tag{10b}$$

Here Ψ is a *section* and A, a *connection* one-form on a $U(1)$ line bundle L over X, ∇_A and ∇_A^* are the covariant derivative and its adjoint, $\Delta_A = \nabla_A^* \nabla_A$, and d and d^* are the exterior derivative and its adjoint, which replace curl and curl*.

Non-Abelian vortex lattices

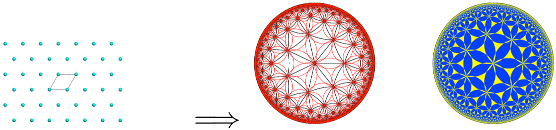
Vortex lattice solutions: from Abelian to non-Abelian lattices.

Lattice λ is an Abelian, discrete group of isometries of $\mathbb{R}^2 \simeq \mathbb{C}$ acting freely on $\mathbb{C}$ by translations $\Longrightarrow$ a non-commutative lattice: a discrete group Γ of isometries of the Poincaré half-plane

$$\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}, \quad \text{with metric} \quad |dz|^2/(\operatorname{Im} z)^2,$$

acting freely on $\mathbb{H}$ by Möbius transforms (Fuchsian group),

$$\gamma z = \frac{az+b}{cz+d} \quad \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subset SL(2,\mathbb{R})\right) \tag{11}$$



Tiling of the Poincaré disc $\simeq \mathbb{H}$ with equilateral triangles

$\Longrightarrow$ Consider the GLE on $\mathbb{H}$ and look for solutions gauge-periodic (equivariant) w.r.to Γ

Lattices and Riemann surfaces

By the key uniformization theorem for Riemann surfaces, a Riemann surface X of genus 1 is torus and can be given as

$$\mathbb{T} = \mathbb{C}/\lambda,$$

where λ is a standard lattice, and of genus ≥ 2 can be given by

$$X = \mathbb{H}/\Gamma,$$

where $\mathbb{H}$ is the Poincaré half-plane and Γ, a Fuchsian group.

$\Gamma-$equivariant functions and vector fields on $\mathbb{H} \Longleftrightarrow$ *sections* and *connections* of the line bundle over the Riemann surface $X = \mathbb{H}/\Gamma$.

For the GLE on an arbitrary RS, X, normal solutions are $(0, A^b)$, with A^b a constant curvature connection. At $b = b_*$, the state $(0, A^b)$ losses stability and a new non-constant curvature, energy minim. soln emerges.

Summary, Discussion and References

We discussed :

- the Ginzburg-Landau equations, playing a key role in the condensed matter and particle physics.
- results on symmetry breaking and emergence of non-transl. invariant ground states which form vortex lattices.
- an extension of the existence results to general Riemann surfaces (replacing the Abelian lattices $\mathcal{L}$ acting on $\mathbb{C}$ by the non-Abelian ones Γ (Fuchsian groups) acting on $\mathbb{H}$).

Extensions. GLE is the simplest ($U(1)$) gauge field theory. Hence, it is natural to extend it to
$U(m), m > 1$ gauge theory - the Yang-Mills-Higgs theory, the bedrock of the particle physics.

Review: I.M. Sigal, Differential equations of quantum mechanics. Quarterly of Applied Mathematics 80 451-480 (2022).

Thank-you for your attention