Quantum vortices and vortex lattices

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Lectures 1-2: Abelian gauge theories and equivariant states

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Objectives

In these lectures, we will consider the phenomenon of formation of quantum vortices and vortex lattices. They appear in all four key partial differential equations (PDE) of condensed matter and particle physics:

(a) The Ginzburg-Landau equations (superconductivity and particle physics);
(b) The Chern-Simons equations (fractional quantum Hall effect and topological QFT);
(c) The Gross-Pitaevskii equation (superfluidity and Bose-Einstein condensation);
(d) The Landau-Lifshitz equation (magnetism).

A remarkable fact fact about these equations is that their solutions are 'quantized': they can be classified by topological invariants taking integer values.

Abelian gauge theories

An important feature unifying the GLE and CSE is that both present the key and only example of gauge theories. The CSE could be abelian or non-abelian, the GLE is assumed to be Abelian.

The non-Abelian generalization of the GLE is called the Yang-Mills-Higgs system.

In the first two lectures, I will present some generalities about the pure Abelian gauge theories and their coupling to matter. There are two such theories:

- the Maxwell theory in $4D$,
- the Chern-Simons (CS) theory in $3D$.

Principle of minimal action

I will use that a physical theory could be given by a space, X, of states, $u \in X$, and an action functional, $S(u)$, defined on paths, $u(t) : [0, T] \to X$, in this space, with the evolution equations given by the principle of minimal action, i.e. by the Euler–Lagrange equations

$$\delta_u S(u) = 0,$$

where $\delta_u S(u)$ denotes the Gâteaux (variational) derivative:

$$\delta_u S(u)\,\xi = \frac{d}{d\lambda} S(u + \lambda\xi)\big|_{\lambda=0},$$

or the L^2-gradient $\nabla_u S(u)$ defined by

$$\langle \nabla_u S(u), \xi \rangle = \delta_u S(u)\,\xi.$$

Gross–Pitaevskii (GP) action

For example, we consider the superfluids and Bose-Einstein condensates. In the 'mean-field' approximation, their states are described by an order parameter $\psi : \mathbb{R}^{d+1} \to \mathbb{C}$ and the dynamics are given by Gross–Pitaevskii (GP) action

$$S(\psi) = -\int_{\mathbb{R}^{d+1}} \left(\mathrm{Im}(\bar{\psi}\,\partial_{\mathrm{t}}\psi) + |\nabla\psi|^2 + \mathrm{V}|\psi|^2 + \mathrm{G}(|\psi|^2)\right) \mathrm{d}\mathrm{x}\,\mathrm{d}\mathrm{t},$$

where G is the self-interaction term, specifically, $G(|\psi|^2) = \frac{1}{2}\kappa|\psi|^4$. The Euler–Lagrange equation for this action yield the GPE

$$i\frac{\partial\psi}{\partial t} = H\psi + g(|\psi|^2)\psi, \tag{1}$$

where $H := -\Delta + V$ and $g(s) = G'(s)$.

$U(1)$-gauge theories: generalities

A gauge theory is defined by an action on gauge fields A [1], invariant under the corresponding gauge transformations.

It is coupled to a matter field through the covariant derivatives induced by these gauge fields (connections).

Gauge field. We can think of A as a co-vector field[2]

$$A = (a_0, a_1, \ldots, a_d),$$

on the $(d+1)$-dimensional space-time $\mathbb{R}^{d+1}$, with values in the Lie algebra $\mathfrak{g}$ of a Lie group G. In the Abelian case, $G = U(1)$ and $\mathfrak{g} = i\mathbb{R}$. So we change A to $-iA$, with the new A real.

[1] connections on a principal bundle

[2] or a one-form $A = -a_0\, dx^0 + a_1\, dx^1 + \cdots + a_d\, dx^d$

Gauge transformations. The action of the local gauge group $C^1(\mathbb{R}^{d+1}, U(1))$ on A is

$$A(x) \mapsto A(x) - i(dg(x))g^{-1}(x), \qquad g \in C^1(\mathbb{R}^{d+1}, U(1)),$$

where $dg := g^\mu_\nu \partial_{x^\nu} g$, where g^μ_ν is the Minkowski metric and $\mu, \nu = 0, \ldots, d$ (the summation over repeated indices is assumed).

Faraday tensor (curvature): $F \equiv F_A := dA \equiv \operatorname{curl} A$, with the components

$$F_{\mu\nu} := \partial_\mu a_\nu - \partial_\nu a_\mu, \quad \mu, \nu = 0, \ldots, d.$$

Let $a = (a_1, \ldots, a_d)$ and $i, j = 1, \ldots, d$. Then the $(0, i)$-part $-\partial_t a - \nabla a_0$ is the electric field, while the (i, j)-part $\operatorname{curl} a$ is the magnetic field.

Hence a and a_0 are identified with the magnetic and electric potentials.

Maxwell and Chern-Simons actions

Maxwell action. For the Maxwell theory, $d = 3$:

$$S_{\mathrm{EM}}(A) := \frac{1}{2} \int_{\mathbb{R}^4} F^{\mu\lambda} F_{\mu\lambda} \, dx \, dt,$$

where $F^{\mu\lambda} := g^{\mu\nu} g^{\lambda\sigma} F_{\nu\sigma}$, or, in terms of the magnetic and electric potentials (omitting the differentials $dxdt$),

$$S_{\mathrm{EM}}(A) = \int_{\mathbb{R}^4} \left(|\partial_t a + \nabla a_0|^2 - |\operatorname{curl} a|^2 \right)$$

Chern–Simons action. For the CS theory, $d = 2$:

$$S_{\mathrm{CS}}(A) := \frac{\kappa}{2} \int_{\mathbb{R}^3} A \cdot \operatorname{curl} A = \frac{\kappa}{2} \int_{\mathbb{R}^3} \varepsilon_{\mu\nu\rho} a_\mu \partial_\nu a_\rho.$$

Both actions are invariant under the gauge transformation

$$(a, a_0) \mapsto (a + \nabla_x \chi, \ a_0 - \partial_t \chi).$$

Matter coupled to a gauge field

We assume that the matter is described by an action functional $S_{\mathrm{matter}}(\psi)$, where ψ runs over states of matter.

To couple the matter field ψ to a gauge field A, we use the principle of minimal coupling: replace the usual derivatives by the covariant ones: $d \to d_A := d + iA$, or

$$\partial_t \to \partial_{t,a_0} := \partial_t - ia_0, \qquad \nabla \to \nabla_a := \nabla + ia,$$

and add either the Maxwell or CS action:

$$S(\psi, A) := S_{\mathrm{matter}}(\psi, A) + S_{\mathrm{gauge}}(A),$$

where

$$S_{\mathrm{matter}}(\psi, A) := S_{\mathrm{matter}}(\psi)\Big|_{d \mapsto d_A}.$$

The Euler–Lagrange equations are

$\delta_\psi S_{\mathrm{matter}}(\psi, A) = 0,\ \delta_A S_{\mathrm{gauge}}(A) = J,\ $ with $\ J := -\delta_A S_{\mathrm{matter}}(\psi, A).$

Coupling to the Gross–Pitaevskii matter field

For matter, we consider the Gross–Pitaevskii (GP) action

$$S_{\mathrm{GP}}(\psi) = -\frac{1}{2}\int_{\mathbb{R}^{d+1}} \left(\mathrm{Im}(\bar{\psi}\,\partial_{\mathrm{t}}\psi) + |\nabla\psi|^2 + \mathrm{G}(|\psi|^2)\right) \mathrm{dx}\,\mathrm{dt},$$

where G is the self-interaction term, specifically, $G(|\psi|^2) = \frac{1}{2}\kappa|\psi|^4$ and $V = 0$, for simplicity. Then the total action for $d = 3$ is

$$S(\psi, A) := S_{\mathrm{GP}}(\psi, A) + S_{\mathrm{gauge}}(A), \tag{2}$$

where $A = (a, a_0)$ and

$$S_{\mathrm{GP}}(\psi, A) := -\frac{1}{2}\int_{\mathbb{R}^4} \left(\mathrm{Im}(\bar{\psi}\,\partial_{\mathrm{t,a_0}}\psi) + |\nabla_{\mathrm{a}}\psi|^2 + \mathrm{G}(|\psi|^2)\right).$$

The Euler–Lagrange equations for ψ and A are

$$\nabla_\psi S_{\mathrm{GP}}(\psi, A) = 0, \qquad \nabla_A S_{\mathrm{gauge}}(A) = J,$$

where

$$J := -\nabla_A S_{\mathrm{GP}}(\psi, A).$$

Computation of the Gâteaux derivatives

Recalling the definition the L^2-gradient $\langle \nabla_u S(u), \xi \rangle = \delta_u S(u)\, \xi$.
where $\delta_u S(u, v)\, \xi$ is the Gâteaux derivative,

$$\delta_u S(u, v)\, \xi = \frac{d}{d\lambda} S(u + \lambda \xi, v)\big|_{\lambda=0},$$

we compute the L^2-gradient:

$$\nabla_\psi S_{\mathrm{GP}}(\psi, A) = H_A \psi + g(|\psi|^2)\psi,$$

$$\nabla_A S_{\mathrm{GP}}(\psi, A) = -(\mathrm{Im}(\bar{\psi}\, \nabla_{\mathrm{a}} \psi), |\psi|^2),$$

$$\nabla_A S_{\mathrm{EM}}(A) = (\partial_t E - \mathrm{curl}^* B, \mathrm{div}\, E),$$

$$\nabla_A S_{\mathrm{CS}}(A) = \kappa\, \mathrm{curl}\, A,$$

Here
$A = (a, a_0),\ H_A := -\Delta_a + a_0,\ -\Delta_a := \nabla_a^* \nabla_a,\ g(s) = G'(s).$

Coupling GP and electromagnetic field

For the GP field coupled to the EM, the Euler–Lagrange equations are

$$i\frac{\partial \psi}{\partial t} = H_A\psi + g(|\psi|^2)\psi, \tag{3a}$$

$$\partial_t(\partial_t a + \nabla a_0) = -\operatorname{curl}^*\operatorname{curl} a - \operatorname{Im}(\bar{\psi}\,\nabla_{\mathrm{a}}\psi), \tag{3b}$$

$$-\Delta a_0 = |\psi|^2. \tag{3c}$$

where $H_A := -\Delta_a + a_0, \quad -\Delta_a := \nabla_a^*\nabla_a, \quad g(s) = G'(s)$. Above,

$$j := \operatorname{Im}(\bar{\psi}\,\nabla_{\mathrm{a}}\psi) \;\text{ and }\; |\psi|^2$$

are the electric current and the charge density. (3) is the Schrödinger-Maxwell system. (The last two equations, (3b) and (3c), are Ampère's and Gauss' laws.)

Coupling to the CS field

Similar computation for the CS field coupled to the GP one, we have

$$i\partial_{t,a_0}\psi = H_A\psi + g(|\psi|^2)\psi, \tag{4a}$$

$$\kappa B = |\psi|^2, \quad -\kappa * E = \mathrm{Im}(\bar{\psi}\,\nabla_{\mathrm{a}}\psi), \tag{4b}$$

where B and E are the magnetic and electric CS fields:

$$B = \operatorname{curl} a \;\text{ and }\; E = -\nabla a_0 - \partial_t a$$

and $*(v_1, v_2) = (-v_2, v_1)$, the Hodge star operator.

(4) is the Schrödinger-Chern-Simons system. Note:
(4b) $\iff \kappa \operatorname{curl} A = J$

Symmetries and equivariance

By a symmetry of the SM– and SCS–matter systems we mean any transformation T : solutions $(\psi, A) \to$ solutions $T(\psi, A)$.

In addition to translations and rotations, the systems are invariant under the local gauge transformations

$$T_\chi^{\mathrm{gauge}} : (\psi, a, a_0) \mapsto \left(e^{i\chi}\psi,\ a + \nabla_x \chi,\ a_0 - \partial_t \chi \right),$$

for sufficiently regular $\chi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$.

Gauge fixing examples:

- Coulomb gauge: $\operatorname{div} a = 0$,
- temporal gauge: $a_0 = 0$,
- Lorenz/radiation gauge: $\operatorname{div} a + \partial_t a_0 = 0$.

For simplicity, we now consider stationary symmetries.

Stationary symmetries

In the stationary case, the SM and SCS systems are invariant under the following transformations.

Gauge transformations:

$$T_\gamma^{\text{gauge}} : (\psi(x), a(x)) \mapsto \left(e^{i\gamma(x)}\psi(x),\ a(x) + \nabla\gamma(x)\right).$$

Translations:

$$T_h^{\text{trans}} : (\psi(x), a(x)) \mapsto (\psi(x+h),\, a(x+h))\,.$$

Rotations:

$$T_\rho^{\text{rot}} : (\psi(x), a(x)) \mapsto \Big(\psi(\rho^{-1}x),\ \rho^{-1}a(\rho^{-1}x)\Big)\,.$$

The time-dependent version includes

$$a_0(x,t) \mapsto a_0(x,t) + \partial_t\gamma(x,t).$$

Equivariant solutions

Let G be a subgroup of either the group of translations or the group of rotations, and let T_g, $g \in G$, be the induced action on the space of solutions:

$$(T_g u)(x) = u(g^{-1}x).$$

A function u satisfying, for some $\chi_g(x), g \in G, x \in M^{d+1}$,

$$T_g u = T^{\mathrm{gauge}}_{\chi_g} u$$

is called G-gauge-invariant, or G-equivariant.

Since T_g is a group representation, the phases χ_g satisfy the cocycle relation

$$e^{i\chi_{gh}} = \left(T_g e^{i\chi_h}\right) e^{i\chi_g},$$

equivalently,

$$\chi_{gh} - T_g\chi_h - \chi_g \in 2\pi\mathbb{Z}.$$

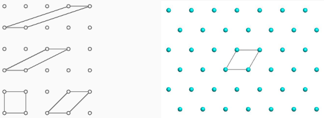
Group of lattice translations

From now on, we fix the dimension: $d = 2$. We will consider two basic groups: lattice translations, $\mathcal{L}$, and rotations, $O(2)$. $\mathcal{L}$ is treated in this lecture and $O(2)$, in lecture 2.

For any basis $\{\nu_1, \nu_2\}$ in $\mathbb{R}^2$, we define the Bravais lattice:

$$\mathcal{L} = \{m\nu_1 + n\nu_2 : m, n \in \mathbb{Z}\} \subset \mathbb{R}^2.$$

A basis defining a lattice $\mathcal{L}$ is not unique. Different bases can define the same lattice.



Lattices and fundamental cells

Gauge-lattice-translation invariant solutions

A pair (ψ, a) is $\mathcal{L}$-equivariant if there exist $g_s : \mathbb{R}^2 \to \mathbb{R}$, $s \in \mathcal{L}$, s.t.

$$\psi(x+s) = e^{ig_s(x)}\psi(x), \quad a(x+s) = a(x) + \nabla g_s(x). \tag{5}$$

Since lattice translations form a group, the family $\{g_s\}_{s\in\mathcal{L}}$ obeys

$$g_{s+t}(x) - g_s(x+t) - g_t(x) \in 2\pi\mathbb{Z}.$$

Proposition
Let $g_s(x)$, $s \in \mathcal{L}$, satisfy the cocycle relation. Then, for any basis $\{\nu_1, \nu_2\}$ in $\mathcal{L}$, the quantity

$$c(g_s) = \frac{1}{2\pi}\Big(g_{\nu_2}(x+\nu_1) - g_{\nu_2}(x) - g_{\nu_1}(x+\nu_2) + g_{\nu_1}(x)\Big)$$

is independent of x and of the choice of basis, and is an integer.

$c(g_s)$ is called the *Chern number*. It is equal to the topological invariant (degree) of an associated manifold called the line bundle.

Proof.
Recall that the function $g_s, s \in \mathcal{L}, x \in \mathbb{R}^2$, obeys

$$g_{s+t}(x) - g_s(x+t) - g_t(x) \in 2\pi\mathbb{Z}.$$

Take here $s = \nu_2, t = \nu_1$, $g_{\nu_2}(x+\nu_1) + g_{\nu_1}(x) - g_{\nu_1+\nu_2}(x) \in 2\pi\mathbb{Z}$, or $s = \nu_1, t = \nu_2$: $g_{\nu_1}(x+\nu_2) + g_{\nu_2}(x) - g_{\nu_1+\nu_2}(x) \in 2\pi\mathbb{Z}$. Subtracting the second relation from the first yields $c(g_s)$, which shows that $c(g)$ is independent of x and is an integer. $\square$

Flux quantization

Theorem. If a satisfies $a(x+s) = a(x) + \nabla g_s(x)$ (the second relation in (5)), then the magnetic flux through a lattice cell Ω is quantized:

$$\frac{1}{2\pi} \int_\Omega \operatorname{curl} a = c(g_s) \in \mathbb{Z}, \tag{6}$$

where $c(g_s)$ is the Chern number.

Eq. (6) relates a geometric quantity (flux of curvature) to a topological one (Chern number $c(g_s)$).

Proof. By Stokes' theorem,

$$\int_\Omega \operatorname{curl} a = \int_{\partial\Omega} a.$$

Using $a(x+s) = a(x) + \nabla g_s(x)$, the boundary integral reduces to

$$g_{\nu_2}(\nu_1) - g_{\nu_2}(0) - g_{\nu_1}(\nu_2) + g_{\nu_1}(0),$$

which is exactly $2\pi c(g_s)$. □

Average flux and size of lattice

Equation (6) implies the relation between the average magnetic flux per lattice cell,

$$b = \frac{1}{|\Omega|} \int_\Omega \operatorname{curl} a,$$

and the area of the fundamental cell:

$$b = \frac{2\pi n}{|\Omega|}. \tag{7}$$

Abrikosov vortex lattice states

Lemma: Physical Periodicity
A state (ψ, a) is $\mathcal{L}$-equivariant iff its physical properties,

- Particle density $n = |\psi|^2$
- Magnetic field $B = \text{curl } a$
- Current density $j = \text{Im}(\bar{\psi}\nabla_a\psi)$

are periodic w.r.t. $\mathcal{L}$.

Such solutions are called *Abrikosov vortex lattice states*, or simply *vortex lattices*.

Proof. If state (ψ, a) satisfies (5), then one can easily check (an exercise) that the associated physical quantities, $|\psi|^2$, $B := \operatorname{curl} a$ and $j := \operatorname{Im}(\bar{\psi}\nabla_a\psi)$ are $\mathcal{L}-$periodic.

In the opposite direction, if $|\psi|^2$, $B := \operatorname{curl} a$ and $j := \operatorname{Im}(\bar{\psi}\nabla_a\psi)$ are periodic with respect to a lattice $\mathcal{L}$, then

$$a(x+s) = a(x) + \nabla\tilde{g}_s(x),$$

for some functions $\tilde{g}_s(x)$. Write $\psi(x) = |\psi(x)|e^{i\gamma(x)}$. Since

$$|\psi(x)| \ \text{ and } \ j(x) = |\psi(x)|^2(\nabla\gamma(x) - a(x))$$

are periodic w.r.to $\mathcal{L}$, we have that $\nabla\gamma(x+s) = \nabla\gamma(x) + \nabla\tilde{g}_s(x)$, which implies that $\gamma(x+s) = \gamma(x) + g_s(x)$, where $g_s(x) = \tilde{g}_s(x) + c_s$, for some constants c_s. To sum up: $\psi(x+s) = e^{ig_s(x)}\psi(x)$ & $a(x+s) = a(x) + \nabla g_s(x)$. $\square$

Thank you for your attention

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Lecture 2: Vortices

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Matter coupled to an Abelian gauge fields

In this lecture, we continue our study of the key class of solutions - equivariant solutions - of matter coupled to an Abelian gauge fields.

The latter is described by the action functional

$$S(\psi, A) := -\int_{\mathbb{R}^4} \left(2\mathrm{Im}(\bar{\psi}\,\partial_{\mathrm{t,a_0}}\psi) + |\nabla_a\psi|^2 + G(|\psi|^2)\right) dx\,dt \\ + S_{\mathrm{gauge}}(A), \tag{8}$$

where $A = (a, a_0)$, G is the self-interaction term and

$$\partial_{t,a_0} := \partial_t - ia_0, \qquad \nabla_a := \nabla + ia.$$

Equivariant solutions

For G (a subgroup of) either the group of translations or the group of rotations, we defined equivariant solutions, as $u = (\psi, a)$ satisfying

$$T_g u = T^{\text{gauge}}_{\chi_g} u,$$

for some $\chi_g(x), g \in G, x \in \mathbb{R}^{d+1}$, where

$$(T_g u)(x) = u(g^{-1}x).$$

Since T_g is a group representation, the phases χ_g satisfy the cocycle relation

$$e^{i\chi_{gh}} = (T_g e^{i\chi_h})\, e^{i\chi_g},$$

equivalently,

$$\chi_{gh} - T_g \chi_h - \chi_g \in 2\pi\mathbb{Z}.$$

Rotation-equivariant solutions

In the last lecture, we considered the *lattice-translation-equivariant static solutions. Now, we consider the rotation-equivariant solutions, i.e. pairs* $u = (\psi, a)$ *satisfying*

$$T_g u = T^{\text{gauge}}_{\chi_g} u, \tag{9}$$

with $g \in G, G = SO(2)$ *and* $(T_g u)(x) = u(g^{-1}x)$. *We can write* $g \in SO(2)$ *as a counterclockwise rotation,* R_α*, in* $\mathbb{R}^2$ *through the angle* α. *Hence the general co-cycle condition* $\chi_{gh} - T_g \chi_h - \chi_g \in 2\pi\mathbb{Z}$ *becomes*

$$\chi_{\alpha+\beta} - \chi_\beta - \chi_\alpha \in 2\pi\mathbb{Z}$$

and therefore $\chi_\alpha(x) = 2\pi n\alpha$*, for some* $n \in \mathbb{Z}$. *Hence* (9) *becomes*

$$\psi(R_\alpha x) = e^{in\alpha}\psi(x),\ a(R_\alpha x) = R_\alpha a(x) \tag{10}$$

(only the global gauge invariance survives.)

Eq. (10) determines the form of (ψ, a):

Lemma (Form of rotation-equivariant states)
Functions $\psi(x)$ and $a(x)$ satisfying (10) *and* $\operatorname{div} a(x) = 0$ *have quantized are eigenfunctions of the spin (angular momentum)*

$$L\psi = n\psi \quad \textit{and} \quad \hat{L}a = Ja, \tag{11}$$

where $L := x\partial_y - y\partial_x, \hat{L} := L\mathbf{1}$ *and* $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, *and are of the form*

$$\psi^{(n)}(x) = f_n(r)e^{in\theta} \quad \textit{and} \quad a^{(n)}(x) = \alpha_n(r)\nabla(n\theta), \tag{12}$$

for some radial functions $f_n(r)$ *and* $a_n(r)$, *where* (r, θ) *are the polar coordinates of* $x \in \mathbb{R}^2$.

Proof. By the first equation in (12), the function $e^{-in\theta}\psi^{(n)}(x)$ is rotation invariant and therefore depends only on $|x|$.

By $\operatorname{div} a(x) = 0$, the vector field $a(x)$ can be written as $a(x) = \alpha(x)x^{\perp}/|x|$. This, by the second equation in (12), shows that the function $\alpha(x)$ is rotation invariant and therefore depends only on $|x|$. $\square$

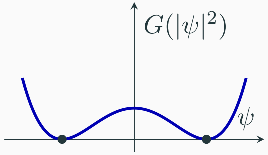
Systems with non-zero particle density

In condensed matter physics, one addresses systems which have non-zero density in the thermodynamic regime.

In the mean-field (Bogolubov) regime such systems are described by PDE of the SM and SCS type but with the self-interaction term $G(|\psi|^2)$ given by the double-well potential:

$$G(|\psi|^2) = \frac{\kappa^2}{4}(|\psi|^2 - 1)^2,$$

or, more generally, $G(s) \geq 0$, with two minima at $\pm a$, with $G(\pm a) = 0$.



Condensates

With this self-interaction term, the SM and SCS systems are called the Ginzburg-Landau and Chern-Simons (-Ginzburg-Landau) systems.

Such systems have the homogeneous stationary solutions satisfying

$$|\psi(x)| \equiv 1, \qquad \operatorname{curl} A(x) = \text{const}.$$

Such solutions are called the condensates.

Localized solutions

Consider stationary states which are local perturbations of the condensate $\psi(x) : |\psi(x)| = 1$, i.e. states satisfying

$$\lim_{|x|\to\infty} |\psi(x)| = 1.$$

For such states, we can define the winding number

$$\deg(\psi) := \frac{1}{2\pi i} \int_{|x|=R} \psi^{-1} \nabla \psi, \tag{13}$$

for $R \gg 1$, s.t. $\inf_{x:|x|=R} |\psi(x)| \geq \delta > 0$. $\deg(\psi)$ is independent of R, as long as $\inf_{x:|x|=R} |\psi(x)| \geq \delta > 0$. In fact, we can replace the circle $|x| = R$ in (13) by any other large contour γ satisfying $\inf_{x \in \mathrm{Ran}\ \gamma} |\psi(x)| \geq \delta > 0$. Moreover, $\deg(\psi)$ is an integer.

$\deg(\psi)$ the topological degree of the map

$$\frac{\psi}{|\psi|}\Big|_{|x|=R} : \mathbb{S}^1 \to \mathbb{S}^1, R \gg 1.$$

Magnetic flux quantization

For each state as above we have the quantization of magnetic flux.

Theorem (Magnetic flux quantization)

Assume (i) $\lim_{|x|\to\infty} |\psi(x)| = 1$ *and (ii)* $|\nabla_a \psi(x)| \lesssim |x|^{-\alpha}, \alpha > 1$. *Then*

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} \operatorname{curl} a(x) = 2\pi \deg(\psi) \in \mathbb{Z}. \tag{14}$$

Proof. Let D_R be a disc of radius R centred at the origin. By Stokes theorem, we have $\int_{D_R} \operatorname{curl} a(x) = \int_{\partial D_R} a(x)$. Next, we estimate

$$\int_{\partial D_R} |a(x) - (\frac{1}{i\psi}\nabla\psi)(x)| = \int_{\partial D_R} |(\frac{1}{i\psi}\nabla_a\psi)(x)| \lesssim R^{-\alpha}.$$

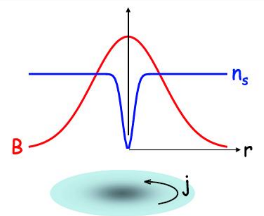
Writing $a - \frac{1}{i\psi}\nabla\psi = -\frac{1}{i\psi}\nabla_a\psi$, using the estimate above and taking $R \to \infty$ and using that $\int_{\partial D_R}(\frac{1}{\psi}\nabla\psi)(x) = \deg(\psi)$ is independent of R, we conclude that (14) holds. $\square$

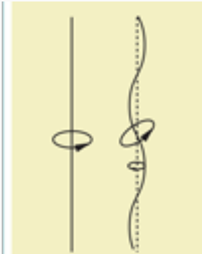
Topological sectors. We say a pair, $u = (\psi, a)$, satisfying (i) and (ii), has the degree n, if $\deg(\psi) = n$ and $\int_{\mathbb{R}^2} \operatorname{curl} a(x) = 2\pi n$. Now, we can classify such pairs by the topological degree and form topological sectors. In the time-dependent setting, this leads to the topological conservation laws.

For rotation, equivariant solutions, $(\psi^{(n)}, a^{(n)})$, found in the lemma above, we have $\deg(\psi^{(n)}) = n$.

The pair $(\psi^{(n)}, a^{(n)})$ is called the n-*vortex* (*magnetic* or *Abrikosov* in the case of superconductors, and *Nielsen-Olesen* or *Nambu string* in the particle physics case).

The next figures show a profile a vortex and its extension to the third dimension (line vortex, or vortex filament)





Thus, we characterized the new type of solutions - vortices - in two different ways: topologically (by degree) and algebraically (by $O(2)$-eqivariance).

Lattice and vortex states

We considered two types of solutions: lattice states and vortex states. They are characterized algebraically by group representations of lattice-translation and rotation groups.

In addition, vortex states are characterized topologically (by degree). It turns out that lattice states are also characterized topologically by degree and this degree is equal to to the Chern number.

A key difference between the rotation and translation equivariant states is that the former are localized and have finite energy, while the latter are extended states ('uniformly spread' over the entire space) and therefore having infinite or zero energy.

We will see later on that lattice states are ground states and vortices are their finite energy excitations (of the corresponding physical systems). We will also see later that lattice states have an interesting shape: they are made of vortices arranged in lattices.

Conclusion

We considered the electro-magnetic (or Yang-Mills) and Chern-Simons gauge theories and their coupling to matter. These are the only Abelian gauge theories.

We classified solutions of these theories by a topological degree (which in the lattice case is equal to the Chern number), and equivariance group representations.

This classification leads to quantization of the matter spin (angular momentum) and of flux of the gauge (magnetic) field.

Group classification (rotation or lattice-translation group) leads their geometrical localization as localized and extended states - vortex states and lattice states - depending on whether the group is compact or not.

The next questions: existence, uniqueness and stability of these states. To address these questions, we have to turn to PDEs which describe them - the Ginzburg-Landau and Chern-Simons systems. This will be done in the next two lectures.

Gauge equivalence and physical symmetry

Gauge symmetry played a central role in our analysis above.

Unlike physical symmetries (translation, rotation, etc), solutions related by gauge transformations are physically and geometrically equivalent. Hence it is natural to consider equivalence classes

$$[(\psi, a)] := \{(\psi', a') : \exists \chi,\ (\psi', a') = T^{\text{gauge}}_{\chi}(\psi, a)\}.$$

Choosing a representative from a class is called choosing a gauge.

Let G_{rm} be the group of rigid motions, the semidirect product of translations and rotations. An equivalence class $[u]$ is invariant under a subgroup $G \subset G_{\mathrm{rm}}$ iff

$$[T_g u] = [u], \qquad \forall g \in G,$$

i.e. iff for every $g \in G$, $T_g u$ is gauge-equivalent to u.

Thank you for your attention