Mathematics of the 2D anyon gas

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Topological Textures school, Zagreb, 2026

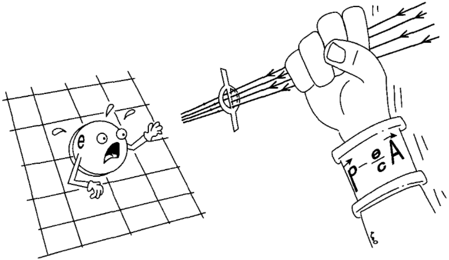


Image credit: D.P. Arovas, in Geometric Phases in Physics (eds. A. Shapere and F. Wilczek), 1989.

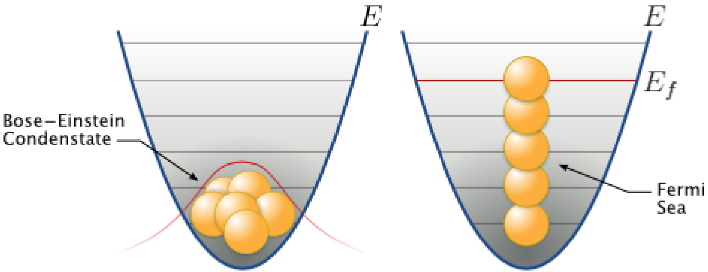
Outline of the course

1. Quantum statistics & transmutation
2. Local exclusion & stability for ideal anyons
3. The almost-bosonic, extended, interacting anyon gas
4. Nonabelian anyons & topological quantum computing
5. Outlooks and further references

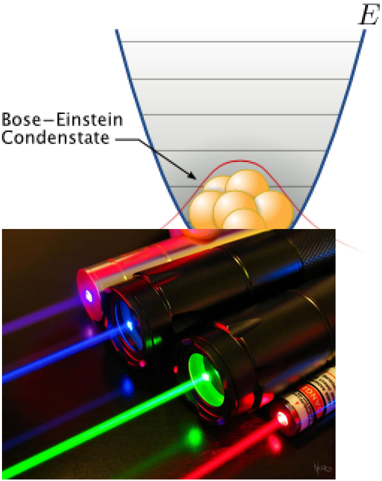
Lecture I: Quantum statistics & transmutation

Aim: terminology & rigorous definition of ideal anyon gas

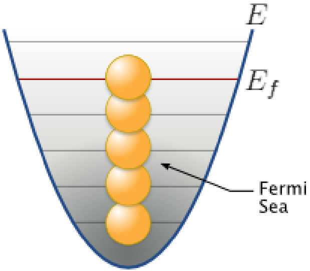
Quantum statistics in 3D



Cold **bosons** Cold **fermions**

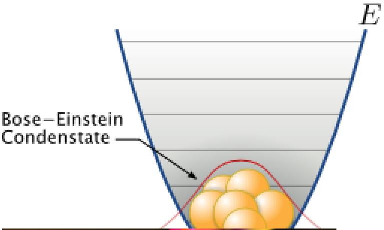


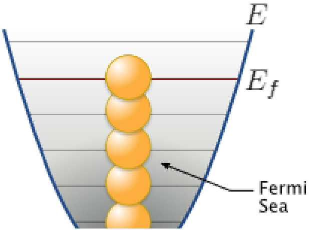
force carriers (coherent/degenerate)



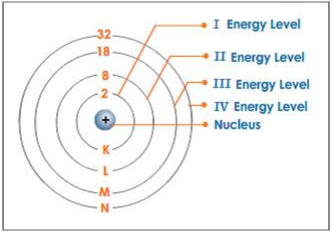
Cold **fermions**

Quantum statistics in 3D





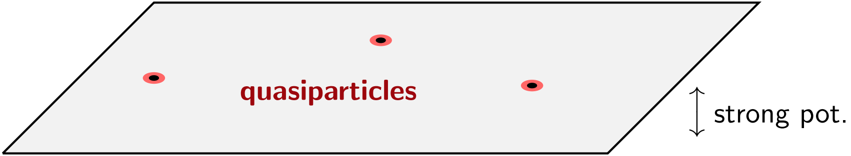
force carriers (coherent/degenerate)



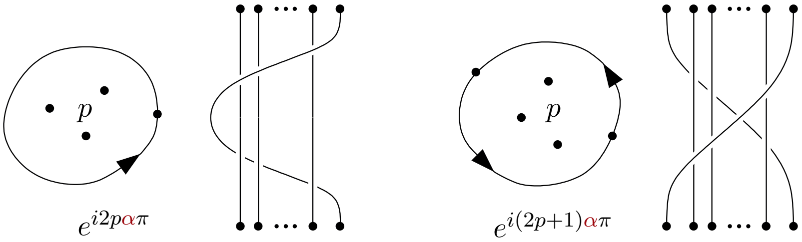
matter (stable/non-degenerate)

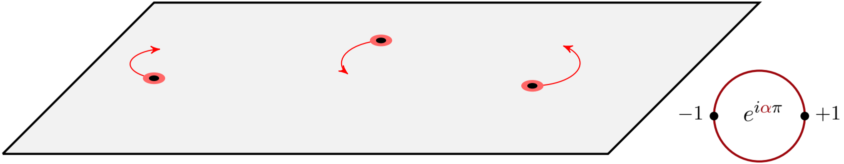
Quantum statistics in 2D

Different in 2D! (and in 1D)



Quantum statistics in 2D (exchange a/symmetry)





$\Psi(\mathbf{x}_2, \mathbf{x}_1) = e^{\pm i\theta}\Psi(\mathbf{x}_1, \mathbf{x}_2)$ $\qquad$ $\theta = \alpha\pi$ **any** phase $\Rightarrow$ "**anyons**"

Anyon (/plekton/nonabelion) models

most computational	most general	most practical
$\downarrow$	$\downarrow$	$\downarrow$
algebraic $\longrightarrow$	**geometric**	$\longleftarrow$ / $\dashrightarrow$ **magnetic**

statistics transmutation

$$\boxed{\rho\colon B_N \to \mathrm{U}(\mathcal{F}_N)}$$

Goldin, Menikoff, Sharp '81,'85

Moore, Seiberg, Witten '89
Fröhlich et al. '88-'90
Kitaev '97-'06-
+Freedman, Wang '02-
Bonderson '07
+Gurarie, Nayak '11
...
DL, Qvarfordt '17,'20

Leinaas, Myrheim '77

Dowker '85
Mueller, Doebner, '93
Mund, Schrader '95
Dell'Antonio, Figari, Teta '97
Goldin, Majid, '04
Maciazek, Sawicki '19
...
Review: DL, Qvarfordt '20/'26

Wilczek '82; Wu '84
+Arovas, Schrieffer '84, +Zee '85

Moore, Read '91, +Rezayi '99
Verlinde '91, Lee, Oh '94 (NACS)
Mancarella, Trombettoni, Mussardo '13
...
DL, Solovej '13,'14
+Rougerie, Larson, Seiringer,
Correggi, Duboscq '15-
+Yakaboylu et al '19-
+Lambert '23
+Ataei, Nguyen '24, +Girardot '25

History: Gibbs

1. Particle Statistics in Quantum Mechanics

1.1. The notion of identical particles and their statistics has played a key role in the study of physical systems since the early days of statistical mechanics and quantum theory.

Consider a classical system of n *identical* particles confined to some bounded region Λ of physical space. Let Γ_n denote the classical phase space of the system, and let H_n be its classical Hamilton function. Also, let T denote the physical temperature, let k denote Boltzmann's constant, let $\beta = (kT)^{-1}$, and let $\rho = n/|\Lambda|$ be the density of the system, with $|\Lambda|$ the volume of Λ.

According to Gibbs, the Helmholtz *free energy*, $F_n(\Lambda, T)$, of the system in thermal equilibrium at temperature T is given by

$$F_n(\Lambda, T) = -kT \ln Z_n(\Lambda, T), \tag{1.1}$$

where the *canonical partition function*, $Z_n(\Lambda, T)$, is given by

$$Z_n(\Lambda, T) = \frac{1}{n!} \int_{\Gamma_n} e^{-\beta H_n(P_n, Q_n)} \frac{dP_n\, dQ_n}{h^{3n}}, \tag{1.2}$$

where (P_n, Q_n) denotes a point in Γ_n, with $Q_n \in \Lambda^{\times n}$, and $dP_n dQ_n$ is the Liouville measure on Γ_n. The factor h^{-3n}, where h is a constant with the dimension of an action, is there to make $Z_n(\Lambda, T)$ dimensionless. The *key factor* expressing the *indistinguishability* of the n particles in the system is the factor "$1/n!$" on the r.h.s. of (1.2). As recognized by Gibbs, this factor is crucial if one wants the free energy to be an *extensive quantity*, i.e., proportional to the volume $|\Lambda|$, at a fixed value of the density ρ of the system. The factor $1/n!$ avoids *overcounting* in the calculation of the canonical partition function: In performing the integral over Γ_n on the r.h.s. of (1.2), particle configurations, (P_n, Q_n), differing from each other only by a permutation of the names of the particles are all included separately, with the *same weight* $\exp[-\beta H_n(P_n, Q_n)]$ (assuming the Hamilton function H_n is invariant under all permutations of particle labellings), although they represent the same physical state of the system, since the particles are indistinguishable. As there are $n!$ permutations of the names, or labellings, of n particles, the factor $1/n!$ on the r.h.s. of (1.2) just fixes the problem of overcounting. This factor and the appearance of the constant h on the r.h.s. of (1.2) are traces of quantum theory within classical statistical mechanics!

The story is similar for the other ensembles of equilibrium statistical mechanics, as discussed by Gibbs.

This is the first instance where the notions of identical particles and their indistinguishability have had important physical consequences. For a review of rigorous results in the context of Gibbsian classical (and quantum) statistical mechanics (thermodynamic functions, equilibrium states, thermodynamic limit, etc.) we refer to Ruelle's book **[1]**.

The role played by the notion of identical particles and their statistics is considerably more dramatic in quantum theory. It became clear shortly after the discovery of quantum theory that the properties of systems of identical particles not only depend on the inter-particle forces, but also on the statistics of the particles (Bose–Einstein or Fermi–Dirac statistics, in a physical space of three (or more) dimensions; Fermi–Dirac statistics reflecting the Pauli principle).

This article is devoted to a review of physical consequences of and mathematical results on statistics in quantum theory, an area that would surely have been of interest to Gibbs, and that has seen quite dramatic developments, during the past one or two decades: Besides the elucidation of consequences of statistics in atomic—condensed matter—and astrophysics, the theory of particle statistics in two-dimensional space has been recognized to be at the basis of a proper theoretical understanding of the fractional quantum Hall effect **[2]** and, perhaps, of certain highly anisotropic high-T_c superconductors **[3]**. Equally important, a detailed mathematical analysis of statistics in (low-dimensional) local quantum theory (quantum field theory) turns out to be deeply related to several, exciting recent developments in pure mathematics, including generalizations of Tannaka–Krein duality theory for compact groups, the theory of Yang–Baxter representations of the braid groups, and of projective representations of the mapping class groups of Riemann surfaces, knot theory, new invariants for three-dimensional manifolds, the theory of quasi-triangular Hopf—and quasi-Hopf algebras (quantum groups)—etc. We hope to give the reader a brief glimpse of these exciting developments.

1.2. In order to explain what particle statistics means in quantum theory, we must briefly review some basic aspects of *quantization*. Let M be the classical configuration space of a physical system. In quantum mechanics, pure states of this system are described by possibly multi-valued wave functions, ψ, on M. More precisely, states correspond to sections of a complex vector bundle, E, with base space M and fibre V, where V is some complex Hilbert space, typically $V = \mathbb{C}$, or $V = \mathbb{C}^N$, $N = 2, 3, \ldots$, and the sections correspond to single-valued functions on $\widetilde{M}$, the universal covering space of M, with values in V and having certain covariance properties under the action of covering transformations: There exists a unitary representation U of the fundamental group $\pi_1(M)$ of M on V such that

$$\psi([\omega]q) = U([\omega])\psi(q), \tag{1.3}$$

for every $q \in \widetilde{M}$ and every $[\omega] \in \pi_1(M)$.

The Hilbert space structure on the space of pure states is given by a positive-definite scalar product which is defined as follows. For ψ and φ two functions on $\widetilde{M}$ satisfying (1.3), we define

$$\langle \psi, \varphi \rangle := \int_{\mathscr{F}} \langle \psi(q), \varphi(q) \rangle_V \, dq, \tag{1.4}$$

History: Leinaas & Myrheim

IL NUOVO CIMENTO Vol. 37 B, N. 1 11 Gennaio 1977

On the Theory of Identical Particles.

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(ricevuto il 16 Agosto 1976)

Summary. — The classical configuration space of a system of identical particles is examined. Due to the identification of points which are related by permutations of particle indices, it is essentially different, globally, from the Cartesian product of the one-particle spaces. This fact is explicitly taken into account in a quantization of the theory. As a consequence, no symmetry constraints on the wave functions and the observables need to be postulated. The two possibilities, corresponding to symmetric and antisymmetric wave functions, appear in a natural way in the formalism. But this is only the case in which the particles move in three- or higher-dimensional space. In one and two dimensions a continuum of possible intermediate cases connects the boson and fermion cases. The effect of particle spin in the present formalism is discussed.

1. – Introduction.

In the quantum description of a system of identical particles, the indistinguishability of the particles has consequences which deeply affect the physical nature of the system. Usually, the indistinguishability is expressed in the theory by imposing symmetry constraints on the state functions and on the observables. Thus, the state functions can be either symmetric or antisymmetric with respect to the interchange of two particle co-ordinates, and all the observables must be invariant under such an operation. The physical consequences of this postulate seem to be in good agreement with the experimental facts. However, the theoretical justification of the postulate, as found, for example, in standard textbooks (1-3), often seems unclear, and several authors have attempted a

(1) A. Messiah: *Quantum Mechanics*, Chap. XIV (Amsterdam, 1962).
(2) L. I. Schiff: *Quantum Mechanics*, Chap. 10 (New York, N. Y., 1968).
(3) E. Merzbacher: *Quantum Mechanics*, Chap. 18 (New York, N. Y., 1961).

History: Goldin, Menikoff & Sharp

Representations of a local current algebra in nonsimply connected space and the Aharonov–Bohm effect [a)]

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A recent paper established technical conditions for the construction of a class of induced representations of the nonrelativistic current group $\mathscr{S} \wedge \mathscr{K}$, where $\mathscr{S}$ is Schwartz's space of rapidly decreasing C^∞ functions, and $\mathscr{K}$ is a group of C^∞ diffeomorphisms of $\mathbb{R}^s$. Bose and Fermi N-particle systems were recovered as unitarily inequivalent induced representations of the group by lifting the action of $\mathscr{K}$ on an orbit $\Delta \subseteq \mathscr{S}'$ to its universal covering space $\widetilde{\Delta}$. For $s \geqslant 3$, $\widetilde{\Delta}$ is the coordinate space for N particles, which is simply connected. In two-dimensional space, however, the coordinate space is multiply connected, implying induced representations other than those describing the usual Bose or Fermi statistics; these are explored in the present paper. Likewise the Aharonov–Bohm effect is described by means of induced representations of the local observables, defined in a nonsimply connected region of $\mathbb{R}^s$. The vector potential plays no role in this description of the Aharonov–Bohm effect.

PACS numbers: 03.65.Bz

I. INTRODUCTION

Nonrelativistic quantum mechanics can be described by means of the local operators $\rho(\mathbf{x})$, the number density of particles, and $\mathbf{J}(\mathbf{x})$, the particle flux. When integrated with test functions having components in Schwartz' space $\mathscr{S}$ (C^∞ functions of rapid decrease), these operators form a Lie algebra. We define $\rho(f) = \int \rho(\mathbf{x}) f(\mathbf{x})\, d\mathbf{x}$ and $J(\mathbf{g}) = \int \mathbf{J}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x})\, d\mathbf{x}$; then the commutation relations (at fixed time) become

$$[\rho(f_1), \rho(f_2)] = 0, \quad (1.1)$$

$$[\rho(f), J(\mathbf{g})] = i\rho(\mathbf{g}\cdot\nabla f), \quad (1.2)$$

$$[J(\mathbf{g}_1), J(\mathbf{g}_2)] = iJ([\mathbf{g}_1, \mathbf{g}_2]), \quad (1.3)$$

where $[\mathbf{g}_1, \mathbf{g}_2] = \mathbf{g}_2 \cdot \nabla \mathbf{g}_1 - \mathbf{g}_1 \cdot \nabla \mathbf{g}_2$ is the Lie bracket of the vector fields $\mathbf{g}_1$ and $\mathbf{g}_2$. Exponentiation of the current commutators leads to the consideration of continuous unitary representations of the semidirect product group $\mathscr{S} \wedge \mathscr{K}$; where $\mathscr{S}$ is Schwartz's space under addition, $\mathscr{K}$ is a group

Δ to its universal covering space $\widetilde{\Delta}$. In this way Bose and Fermi N-particle representations are recovered as induced representations on the same orbit, and it appears that representations describing parastatistics are similarly obtained.[6] Thus the representations of $\mathscr{S} \wedge \mathscr{K}$ depend importantly on the connectedness (more specifically, the homotopy) of the orbit on which the measure is concentrated.

In three or more dimensions, the coordinate space for N particles is simply connected, even after removal of the set in which two particles have the same coordinates. In two-dimensional space, however, the coordinate space is multiply connected, leading to induced representations other than the usual Bose or Fermi representations. These are described in Sec. II of the present paper.

In Sec. III we apply our results to describe the Aharonov–Bohm effect[7] for a single boson or fermion exclusively in terms of observables. The reader who wishes to bypass the mathematical description of induced representations can proceed directly to this section. Excluding the particle from access to the region of nonvanishing magnetic field results in

GROUP DESCRIBING PARTICLES IN TWO-DIMENSIONAL SPACE

For a representation of $\mathscr{S} \wedge \mathscr{K}$ describing N particles in s-dimensional, space, we have the $\mathscr{K}$-orbit

[a)] Work supported by the U.S. Department of Energy.

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Quantum Mechanics of Fractional-Spin Particles

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Composites formed from charged particles and vortices in (2+1)-dimensional models, or flux tubes in three-dimensional models, can have any (fractional) angular momentum. The statistics of these objects, like their spin, interpolates continuously between the usual boson and fermion cases. How this works for two-particle quantum mechanics is discussed here.

PACS numbers: 03.65.Ca, 03.65.Ge, 05.30.-d

In a recent note[1] I showed that charged particles orbiting around magnetic flux tubes have orbital angular momentum integer $+q\Phi/2\pi$; this phenomenon is realized for example in the vortices of a type-II superconductor and in string solutions of gauge theories.[1] Closely related observations were made previously by Hasenfratz,[2] and recently by Goldin and Simon.[3] See also the discussion by Peshkin.[4] If there is a generalized spin-statistics connection, we must expect that the flux-tube–particle composites have unusual statistics, interpolating between bosons and fermions. Since interchange of two of these particles can give *any* phase, I will call them generically anyons.

In this paper some elementary examples in the quantum mechanics of anyons are worked out. Description of these particles requires some widening of the notion of a wave function. Also, we will see that the energy levels of a system of two noninteracting anyons are not in general simply related to the one-anyon levels.

Although practical applications of these phenomena seem remote, I think they have considerable methodological interest and do shed light

One anyon.—Let us recall how the fractional L_z arises. Charged particles orbiting around a flux tube carrying flux Φ are subject to an azimuthal vector potential

$$A_\varphi = \Phi/2\pi r. \tag{1}$$

Although the potential gives vanishing magnetic field strength, and therefore is negligible in classical physics, it does play a role in quantum mechanics.[7] It is convenient to eliminate A_φ by a gauge transformation:

$$A_i' = A_i - \partial_i \Lambda = 0, \quad \Lambda = \Phi\varphi/2\pi. \tag{2}$$

The required Λ is, however, not a well-defined (2π periodic) function of the angle φ. This reflects itself in the transformation of charged-particle wave functions:

$$\psi'(\varphi) = e^{iq\Phi\varphi/2\pi}\psi(\varphi). \tag{3}$$

In fact, since $\psi(\varphi)$ is 2π periodic we find from (3) that

$$\psi'(\varphi + 2\pi) = e^{iq\Phi}\psi'(\varphi). \tag{4}$$

The allowed angular wave functions $\psi'(\varphi) \sim e^{im\varphi}$ therefore have $m =$ integer $+q\Phi/2\pi$, which is to

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Multiparticle Quantum Mechanics Obeying Fractional Statistics

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We obtain the rule governing many-body wave functions for particles obeying fractional statistics in two (space) dimensions. It generalizes and continuously interpolates the usual symmetrization and antisymmetrization. Quantum mechanics of more than two particles is discussed and some new features are found.

PACS numbers: 03.65.Ca, 03.65.Ge, 05.30.–d

In two (space) dimensions, there are allowed to be particles of fractional angular momentum or spin.[1,2] If there is a generalized spin-statistics connection, such particles are expected to have unusual (fractional) statistics which continuously interpolates between the normal bosons and fermions. (An example for such interpolation is known in one dimension.[3]) The intriguing problem of how it works is interesting both from the viewpoint of theoretical principles and from the prospect of physical applications. A possible relevance of fractional statistics to the quantized Hall effect has been recently suggested.[4]

Two simple models have been proposed for particles obeying fractional statistics by Wilczek[1,5] Yang and Yang,[3] and Wilczek and Zee.[6] Two-particle quantum mechanics was analyzed in detail. A low-density expansion of the partition function interpolating the standard statistics was obtained. As pointed out in these papers, Feynman's path-integral formulation is a good starting point. However, the formalism in terms of wave functions may be practically more convenient. An immediate problem is the general rule governing the many-body wave functions, namely how to generalize the usual rule to obtain a continuous interpolation between symmetrization and antisymmetrization. In this note I answer this question by deriving the desired rule in the two models mentioned above. As an application, I discuss the quantum mechanics of three particles, not yet touched in the literature. Some new features are found which are not present in the two-particle case.

Anyons revisited.—Following Wilczek,[5] I denote composites formed from charged particles and magnetic flux tubes as anyons, since their spin

$$\Delta = q\Phi/2\pi = \theta/2\pi \tag{1}$$

can take any real values. Here q is the charge and Φ the flux. That[5] interchange of two anyons leads to a phase $e^{i\theta}$ is an indication of the fractional statistics. We here consider quantum mechanics for more than two anyons.

The Hamiltonian for a charged particle orbiting around a flux tube can be written as

$$H_0 = \frac{1}{2m_q}\left[-i\frac{\partial}{\partial \vec{r}_q} - q\vec{A}(\vec{r}_q - \vec{r}_f)\right]^2 + \frac{1}{2m_f}\left[-i\frac{\partial}{\partial \vec{r}_f} + q\vec{A}(\vec{r}_q - \vec{r}_f)\right]^2. \tag{2}$$

Here we consider the limit in which the size of the flux tube can be neglected. $\vec{r}_q$ and $\vec{r}_f$ are two-dimensional vectors. Let us assume that the flux tube has a finite effective mass m_f in two dimensions. The form (2) has the advantage that the effect of the interaction is confined to the wave function in the relative coordinate. In a regular gauge the vector potential is

$$q\vec{A}(\vec{r}_q - \vec{r}_f) = -q\vec{A}(\vec{r}_f - \vec{r}_q) = (\theta/2\pi)[\vec{n}\times(\vec{r}_q - \vec{r}_f)]/|\vec{r}_q - \vec{r}_f|^2 \tag{3}$$

(with $\vec{n}$ being the unit vector normal to the two-dimensional plane), and the wave function is single-valued everywhere.

Now we proceed to consider n identical anyons and neglect the electrostatic forces between them (i.e., con-

Quantum statistics done 'right'

The **configuration space** of N **distinguishable** particles: $(\mathbb{R}^d)^N$
The configuration space of N **identical** particles in $\mathbb{R}^d$: [Gibbs]

$$\mathcal{C}^N := \Big((\mathbb{R}^d)^N \setminus \triangle\Big) \Big/ S_N \cong \{N\text{-point subsets of } \mathbb{R}^d\}$$

Distinct points by removal of the **diagonals**:

$$\triangle := \{(\mathbf{x}_1, \ldots, \mathbf{x}_N) \in (\mathbb{R}^d)^N : \exists\, j \neq k \text{ s.t. } \mathbf{x}_j = \mathbf{x}_k\}$$

Exchanges of particles are continuous **loops** in $\mathcal{C}^N$:

$$\{\text{loops in } \mathcal{C}^N \text{ modulo homotopy}\} = \pi_1(\mathcal{C}^N) = \begin{cases} 1, & d = 1, \\ B_N, & d = 2, \\ S_N, & d \geq 3. \end{cases}$$

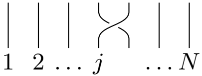
[Leinaas, Myrheim '77; Goldin, Menikoff, Sharp '81; Wilczek '82]

The braid group

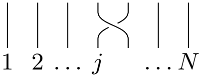
B_N is the **braid group** on N strands:

$$B_N = \left\langle \sigma_1, \ldots, \sigma_{N-1} : \sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1},\ \sigma_j \sigma_k = \sigma_k \sigma_j \right\rangle_{|j-k|>1}$$

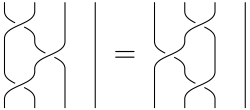
σ_j :



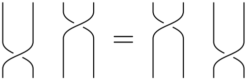
σ_j^{-1} :



Examples in B_4:



$$\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$$



$$\sigma_1 \sigma_3 = \sigma_3 \sigma_1$$

If we add the relations $\sigma_j^2 = 1$ we obtain the **permutation group** S_N

Quantum statistics in $\mathbb{R}^d$ (exchange $\overset{?}{\Rightarrow}$ exclusion)

Ψ wave function for N **distinct** particles in $\mathbb{R}^d$ (diagonals $\triangle$):

$$|\Psi(\mathrm{x})|^2, \qquad \mathrm{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N) \in \mathbb{R}^{dN} \setminus \triangle$$

identical: $\mathbb{R}^d \supset \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\} \in \mathcal{C}^N := (\mathbb{R}^{dN} \setminus \triangle)/S_N$

$$\Psi(\sigma.\mathrm{x}) = \rho(\sigma)\Psi(\mathrm{x}), \qquad \sigma \in \pi_1(\mathcal{C}^N) = 1, B_N \text{ or } S_N$$

$\rho(\sigma)$ **exchange phase** (or operator):

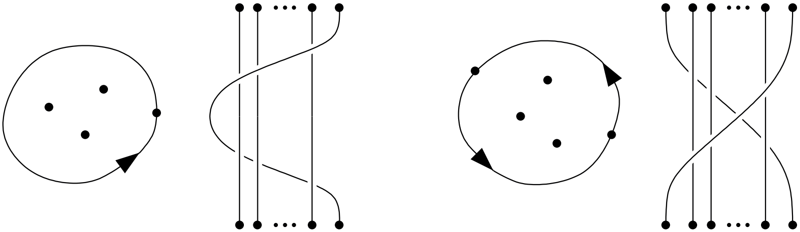
bosons $\rho(\sigma) = +1$, symm., ex. independent identically distributed
$\Psi_0 = \otimes^N u_0 \ \in L^2_{\mathrm{sym}}$

fermions $\rho(\sigma) = \mathrm{sign}(\sigma)$, determinantal correlations & Pauli principle
$\Psi_0 = u_0 \wedge u_1 \wedge \ldots \wedge u_{N-1} \ \in L^2_{\mathrm{asym}}$

anyons ρ unitary rep. of B_N ... **intermediate/fractional statistics?**

Leinaas, Myrheim 1977; Goldin, Menikoff, Sharp 1981; Wilczek 1982

The Hilbert space L^2_α of anyon wave functions



wave function $\Psi\colon \tilde{\mathcal{C}}^N \to \mathbb{C}$, $\langle \Phi, \Psi \rangle = \int_{\mathcal{C}^N} \bar{\Phi}\Psi$

probability $|\Psi|^2\colon \mathcal{C}^N \to \mathbb{R}_+$, $\|\Psi\|^2 = \int_{\mathcal{C}^N} |\Psi|^2$

exchange relations $\boxed{\Psi(\gamma.\tilde{X}) = \rho(\gamma)\Psi(\tilde{X})}$ $\Leftrightarrow$ topological b.c., ex:

$$\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_N) = e^{i\pi\alpha(2p+1)}\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_N)$$

$\alpha \in 2\mathbb{Z}$: **bosons** $\Rightarrow \Psi \in L^2_{\mathrm{sym}}(\mathbb{R}^{2N})$

$\alpha \in 2\mathbb{Z}+1$: **fermions** $\Rightarrow \Psi \in L^2_{\mathrm{asym}}(\mathbb{R}^{2N})$

$\alpha \notin \mathbb{Z}$: **anyons** $\Rightarrow$ multivalued functions

Example: $N = 2$

center of mass $\mathbf{X} := \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2)$
relative coordinate $\mathbf{r} := \mathbf{x}_1 - \mathbf{x}_2 = (r\cos\varphi, r\sin\varphi)$

$$\begin{aligned}
\triangle &= \left\{\mathbf{X} \in \mathbb{R}^2,\ r = 0\right\} \cong \mathbb{R}^2 \\
\Omega &= \left\{\mathbf{X} \in \mathbb{R}^2,\ r > 0,\ 0 \leq \varphi \leq \pi\right\} \subseteq \mathbb{R}^4 \\
\partial\Omega &= \left\{\mathbf{X} \in \mathbb{R}^2,\ r = 0 \text{ or } \varphi = 0 \text{ or } \varphi = \pi\right\} \\
\mathcal{C}^2 &\cong \Omega \text{ with } \mathbf{r} \sim -\mathbf{r} \text{ on } \partial\Omega,\ \text{i.e. } \varphi = 0 \sim \varphi = \pi \\
\tilde{\mathcal{C}}^2 &= \left\{\mathbf{X} \in \mathbb{R}^2,\ r > 0,\ \varphi \in \mathbb{R} \text{ (extended)}\right\}
\end{aligned}$$

exchange relations on Ω (topological b.c.):

$$\Psi(\mathbf{X}, r, \pi) = e^{i\pi\alpha}\Psi(\mathbf{X}, r, 0) \quad \forall\ \mathbf{X} \in \mathbb{R}^2, r > 0$$

exchange relations on $\tilde{\mathcal{C}}^2$ (equivariance) $\Rightarrow L^2_\alpha$:

$$\Psi(\mathbf{X}, r, \varphi{+}n\pi) = e^{i\pi\alpha n}\Psi(\mathbf{X}, r, \varphi) \quad \forall\, \mathbf{X} \in \mathbb{R}^2, r > 0, \varphi \in \mathbb{R}, n \in \mathbb{Z}$$

Kinetic energy (L^2_α is not enough!)

Non-relativistic free kinetic energy: $T = \frac{1}{2m}\sum_{j=1}^N \mathbf{p}_j^2$
quantization: $\hat{\mathbf{p}}_j = -i\hbar\nabla_{\mathbf{x}_j} \Rightarrow$ for **distinguishable** particles:

$$\hat{T}_{\text{dist}} = \underbrace{\frac{\hbar^2}{2m}}_{1}\sum_{j=1}^N(-i\nabla_{\mathbf{x}_j})^2 \quad \text{on} \quad \Psi \in L^2(\mathbb{R}^{2N})$$

For **indistinguishable** particles: use $\hat{T}_{\text{dist}}$ locally on contractible $\Omega \subseteq \mathbb{R}^{2N}$, extend globally on $\tilde{\mathcal{C}}^N$ using the equiv. rel.s / top. b.c. $=: \hat{T}_\alpha$ on L^2_α (locally flat)
Formally, use the **Friedrichs extension** $\Rightarrow$ **ideal anyons**:

$$\langle\Psi, T_\alpha\Psi\rangle = \int_{\mathcal{C}^N}\sum_{j=1}^N |\nabla_{\mathbf{x}_j}\Psi|^2 \geq 0 \quad \text{on } \Psi \in L^2_\alpha$$

For $\alpha = 0$: $T_{\text{sym}} = -\Delta$ on $H^2(\mathbb{R}^{2N}) \cap L^2_{\text{sym}}$ (usual Sobolev space)
For $\alpha = 1$: $T_{\text{asym}} = -\Delta$ on $H^2(\mathbb{R}^{2N}) \cap L^2_{\text{asym}}$

Example: $N = 2$ (exercise)

$$T_\alpha = -\Delta_{\mathbf{x}_1} - \Delta_{\mathbf{x}_2} = \frac{1}{2}(-\Delta_{\mathbf{X}}) + 2 \underbrace{(-\Delta_{\mathbf{r}})}_{-\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\varphi^2}$$

acting on $\Psi \in H^2(\Omega)$ with boundary conditions

$$\Psi(\mathbf{X}, r, \pi) = e^{i\pi\alpha}\Psi(\mathbf{X}, r, 0) \quad \forall\ \mathbf{X} \in \mathbb{R}^2, r > 0,$$

and form domain

$$\langle \Psi, T_\alpha \Psi \rangle = 2 \int_{\mathbf{X}\in\mathbb{R}^2} \int_{r=0}^{\infty} \int_{\varphi=0}^{\pi} \left[\frac{1}{4}|\nabla_{\mathbf{X}}\Psi|^2 + |\partial_r \Psi|^2 + \frac{1}{r^2}|\partial_\varphi \Psi|^2 \right] d\varphi dr d\mathbf{X}$$

Statistics transmutation for $N = 2$ (exercise)

$$\langle \Psi, T_\alpha^{\mathrm{rel}} \Psi \rangle = 2 \int_{r=0}^{\infty} \int_{\varphi=0}^{\pi} \left[|\partial_r \Psi|^2 + \frac{1}{r^2} |\partial_\varphi \Psi|^2 \right] d\varphi dr$$

Write $\Psi(r, \varphi) = e^{i\alpha\varphi} \Phi(r, \varphi)$ where $\Phi(r, \pi) = \Phi(r, 0)$ (bosonic)

Statistics transmutation for $N = 2$ (exercise)

$$\langle \Psi, T_\alpha^{\mathrm{rel}} \Psi \rangle = 2 \int_{r=0}^{\infty} \int_{\varphi=0}^{\pi} \left[|\partial_r \Psi|^2 + \frac{1}{r^2} |\partial_\varphi \Psi|^2 \right] d\varphi dr$$

$$|\partial_r \Phi|^2 \qquad |(\partial_\varphi + i\alpha)\Phi|^2$$

Write $\Psi(r, \varphi) = e^{i\alpha\varphi} \Phi(r, \varphi)$ where $\Phi(r, \pi) = \Phi(r, 0)$ (bosonic)

Statistics transmutation for $N = 2$ (exercise)

$$\langle \Psi, T_\alpha^{\mathrm{rel}} \Psi \rangle = 2 \int_{r=0}^{\infty} \int_{\varphi=0}^{\pi} \left[|\partial_r \Psi|^2 + \frac{1}{r^2} |\partial_\varphi \Psi|^2 \right] d\varphi dr$$

$$|\partial_r \Phi|^2 \qquad |(\partial_\varphi + i\alpha)\Phi|^2$$

Write $\Psi(r, \varphi) = e^{i\alpha\varphi} \Phi(r, \varphi)$ where $\Phi(r, \pi) = \Phi(r, 0)$ (bosonic)

Convenient: $\begin{aligned} \mathbf{r} &= (x, y) && \leftrightarrow z = re^{i\varphi} \in \mathbb{C} \setminus \{0\}, \\ \mathbf{r}^\perp &= (-y, x) && \leftrightarrow iz \end{aligned}$

Let $\Psi = u\Phi$, $u(z) := \frac{z}{|z|} = e^{i\varphi}$, then $\nabla\Psi = u\big(u^{-1}\nabla u + \nabla\big)\Phi$,

$$\mathbf{A} := -iu^{-1}\nabla u = \frac{\mathbf{r}^\perp}{|\mathbf{r}|^2} = \nabla^\perp \log |\mathbf{r}|$$

The vector field $\mathbf{A} \colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$ is closed but not exact:

$$B := dA = \operatorname{curl} \mathbf{A} = \nabla^\perp \cdot \mathbf{A} = \Delta \log |\mathbf{r}| = 2\pi\delta(\mathbf{r})$$

$\Rightarrow$ attachment of 1 unit magnetic flux! (cp. Aharonov-Bohm)

Statistics transmutation in 2D: bosons$\leftrightarrow$fermions

Convenient: $\mathrm{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_N) \;\leftrightarrow\; \mathrm{z} = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N \setminus \triangle$

$$\Psi = U\tilde{\Psi}, \qquad U(\mathrm{z}) := \prod_{j<k} \frac{z_j - z_k}{|z_j - z_k|} = \exp\left(i \sum_{j<k} \arg(z_j - z_k) \right)$$

transmutes $L^2_{\mathrm{sym}} \leftrightarrow L^2_{\mathrm{asym}}$ at the cost of a gauge potential:

$$-i\nabla\Psi = U(-i\nabla + \mathbf{A})\tilde{\Psi}, \quad \mathbf{A}_j(\mathrm{x}) = -iU^{-1}\nabla_{\mathbf{x}_j} U = \sum_{k \neq j} \frac{(\mathbf{x}_j - \mathbf{x}_k)^\perp}{|\mathbf{x}_j - \mathbf{x}_k|^2}$$

$$\hat{T}_{\mathrm{asym}} = \sum_{j=1}^N \hat{\mathbf{p}}_j^2 \qquad \leftrightarrow \qquad \hat{T}_{\mathrm{sym}\to\mathrm{asym}} = \sum_{j=1}^N \left(-i\nabla_{\mathbf{x}_j} + \mathbf{A}_j\right)^2$$

where $\operatorname{curl}_{\mathbf{x}_j} \mathbf{A}_j = 2\pi \sum_{k \neq j} \delta(\mathbf{x}_j - \mathbf{x}_k)$ Aharonov-Bohm fluxes.

Statistics transmutation in 2D: abelian anyons $e^{i\alpha\pi}$

Convenient: $\mathrm{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_N) \;\leftrightarrow\; \mathrm{z} = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N \setminus \triangle$

$$\Psi = U^{\alpha} \tilde{\Psi}, \qquad U(\mathrm{z}) := \prod_{j<k} \frac{z_j - z_k}{|z_j - z_k|} = \exp\left(i \sum_{j<k} \arg(z_j - z_k) \right)$$

transmutes $L^2_{\mathrm{sym}} \leftrightarrow L^2_{\alpha}$ at the cost of a gauge potential:

$$-i\nabla\Psi = U^{\alpha}(-i\nabla + \alpha\mathbf{A})\tilde{\Psi}, \quad \mathbf{A}_j(\mathrm{x}) = -iU^{-1}\nabla_{\mathbf{x}_j} U = \sum_{k \neq j} \frac{(\mathbf{x}_j - \mathbf{x}_k)^{\perp}}{|\mathbf{x}_j - \mathbf{x}_k|^2}$$

$$\hat{T}_{\alpha} = \sum_{j=1}^{N} \hat{\mathbf{p}}_j^2 \qquad \leftrightarrow \qquad \hat{T}_{\mathrm{sym}\to\alpha} = \sum_{j=1}^{N} \left(-i\nabla_{\mathbf{x}_j} + \alpha\mathbf{A}_j\right)^2$$

where $\mathrm{curl}_{\mathbf{x}_j}\, \alpha\mathbf{A}_j = 2\pi\alpha \sum_{k \neq j} \delta(\mathbf{x}_j - \mathbf{x}_k)$ Aharonov-Bohm fluxes.

Lecture II: Local exclusion & stability for ideal anyons

Aim: overview of exchange $\overset{?}{\leftrightarrow}$ exclusion for ideal anyons

Stability of matter in 3D — triumph of QM & math-phys!

$$E_N := \inf \operatorname{spec} \hat{H}_N = \inf \{\mathcal{E}_N[\Psi_N] : \|\Psi_N\|_{L^2} = 1\}$$

1st kind stability: $E_N > -\infty$ for $N = 1$ or for all $N \in \mathbb{N}$
2nd kind stability: $E_N \geq -CN$, as $N \to \infty$

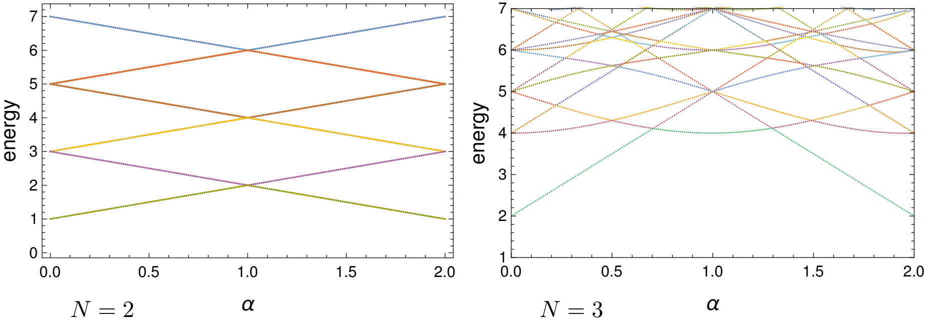
- **Electrogravitics**: Newton 1687; ... Onsager 1939; Fisher, Ruelle '66; Baxter '80
- **Uncertainty principle**: Heisenberg 1925; Schrödinger '26; Hardy '20; Sobolev '38; Ladyzhenskaya, Gagliardo, Nirenberg '58; Ginzburg, Landau '50; Gross, Pitaevskii '61 (NLS)
- **Exclusion principle**: Pauli 1925; Thomas, Fermi 1927; Dyson, Lenard '67; Lieb, Simon '73; Lieb, Thirring '75

Self-generated magnetic fields: $E_N := \inf_{\Psi_N, \mathbf{A}} \mathcal{E}_N[\Psi_N, \mathbf{A}]$
$N = 1$: Fröhlich, Lieb, Loss '86; Loss, Yau '86
$N \to \infty$: Fefferman '95; Lieb, Loss, Solovej '95

Relativity, quantized EM, ... Reviewed in the textbook of Lieb & Seiringer 2010

Exchange vs. exclusion for anyons

How does the exclusion principle generalize to anyons?



$N = 2$ $\qquad$ $N = 3$

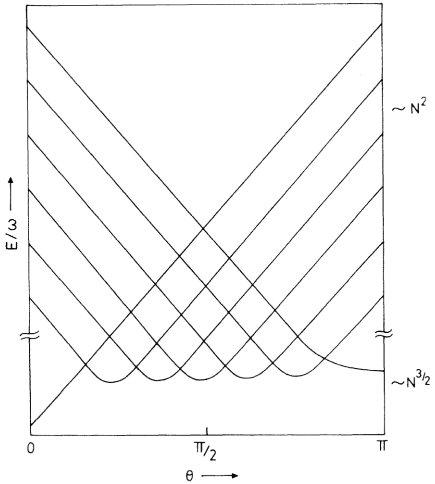
(units of $\hbar\omega$, harmonic trap, center of mass excluded)

Leinaas, Myrheim '77; Wilczek et al. '82,'85
Murthy, Law, Brack, Bhaduri, '91; Sporre, Verbaarschot, Zahed, '91,'92
Canright, Johnson '94: "Fractional statistics: α to β"
Yakaboylu et al. 2019

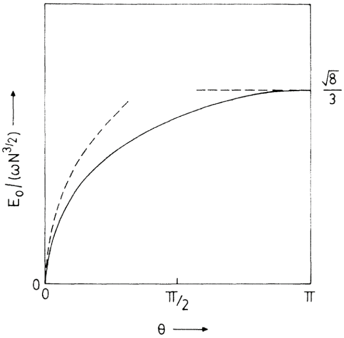
Exactly solvable for $N = 2$, numerics for $N = 3, 4, \ldots$ (very small)

Exchange vs. exclusion for anyons

Potentially interesting dependence on α for $N \gg 1$:



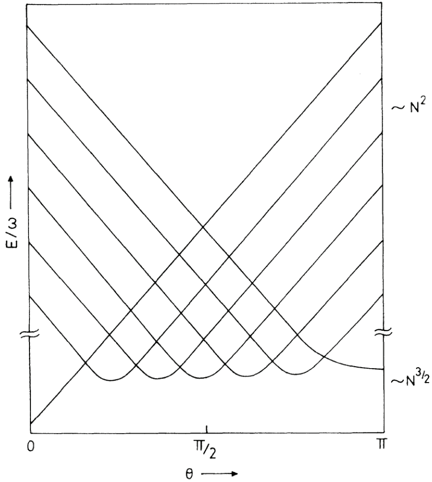
$E \approx 1.43\sqrt{\alpha}E_F$ $\qquad$ $E \approx E_F$



Chitra, Sen, 1992: schematic $N \to \infty$ spectrum, ($\theta = \alpha\pi$, harmonic trap $V = \frac{1}{2}m\omega^2|\mathbf{x}|^2$)

Exchange vs. exclusion for anyons

Potentially interesting dependence on α for $N \gg 1$:

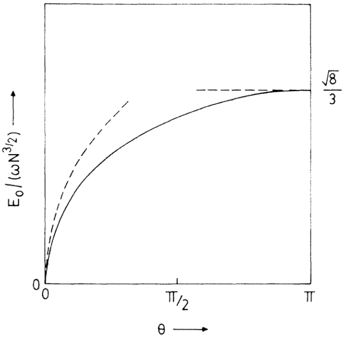


1.54

$E \approx 1.43\sqrt{\alpha}E_F$

ok!

$E \approx E_F$



Chitra, Sen, 1992: schematic $N \to \infty$ spectrum, ($\theta = \alpha\pi$, harmonic trap $V = \frac{1}{2}m\omega^2|\mathbf{x}|^2$)

Girvin et al. '90; Grundberg et al. '91; Manuel, Tarrach '91; Murthy et al. '91:

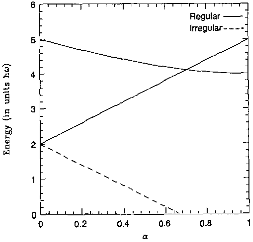


Figure 1. Low lying regular and irregular solutions as a function of α. The linear solutions are exact, while the nonlinear interpolation was obtained numerically. The irregular solutions do not exist for $\alpha \geqslant 2/3$. Note that $\alpha = 0$ is the bosonic end and $\alpha = 1$ is the fermionic end of the spectrum.

Some exact "solutions" with energy $E_\pm(N) = 2N \pm \alpha N(N-1)$:

$$\Psi_\pm \propto \prod_{j<k} |\mathbf{x}_j - \mathbf{x}_k|^{\pm\alpha} e^{-\frac{1}{2}|\mathrm{x}|^2}.$$

Define: As $\mathbf{x}_j - \mathbf{x}_k \to 0$,

$$\Psi_+(\mathrm{x}) \sim |\mathbf{x}_j - \mathbf{x}_k|^{\alpha}$$ "**ideal anyons**"

$$\Psi_-(\mathrm{x}) \sim |\mathbf{x}_j - \mathbf{x}_k|^{-\alpha}$$ "**kreinyons**"

For $N = 2$, compare **Friedrichs** extension vs. **Krein** extension.

See Alonso-Simon '80; Bourdeau, Sorkin '92; Adami, Teta '98; Correggi, Oddis '18; Borrelli, Correggi, Fermi '24; ...

Local exclusion principle

A rigorous and **local approach** to exchange and exclusion.

Statistical repulsion manifests in three ways (at least):

1. effective *scalar* pairwise repulsion $\Rightarrow \Psi \to 0$ at $\triangle$:

$$\hat{T}_\alpha \;\geq\; \frac{4}{N} \max\left\{\alpha_N^2, \frac{\alpha_2^2}{N-1}\right\} \sum_{j<k} \frac{1}{|\mathbf{x}_j - \mathbf{x}_k|^2}$$

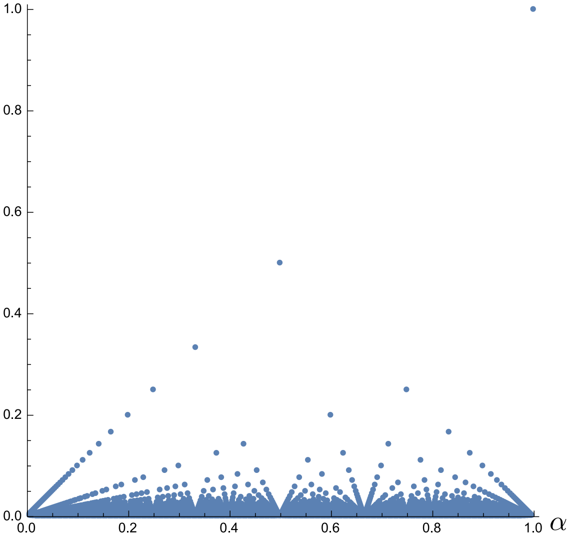
   N-**"fractionality"** of α:

$$\alpha_N := \inf_{p \in \{0,1,\ldots,N-2\}, q \in \mathbb{Z}} \left|(2p+1)\alpha - 2q\right|$$

2. local exclusion principle: $E_N \gtrsim N - 1$
3. degeneracy pressure, ex. Thomas-Fermi or Lieb-Thirring (uncertainty $\leftrightarrow$ exclusion)

Hoffmann-Ostenhof^2, Laptev, Tidblom '08; DL, Solovej '13; DL '17; DL, Qvarfordt '20; Ataei, Girardot, DL '25

The odd-numerator Thomae/"popcorn" function



$$\alpha_{N\to\infty} = \inf_{p,q\in\mathbb{Z}} |(2p+1)\alpha - 2q|$$

Statistical repulsion $\Leftarrow$ Poincaré inequality

Hardy inequality for fermions in $\mathbb{R}^d$: [Hoffmann-Ostenhof[2], Laptev, Tidblom '08]

$$\hat{T}|_{H^1_{\mathrm{asym}}(\mathbb{R}^{dN})} \geq \frac{d^2}{N} \sum_{1 \leq j < k \leq N} \frac{1}{|\mathbf{x}_j - \mathbf{x}_k|^2}$$

Poincaré for fermions: $u(-\omega) = -u(\omega)$, $\omega \in \mathbb{S}^{d-1}$ relative angles

$$\int_{\mathbb{S}^{d-1}} |\nabla_\omega u|^2 \, d\omega \ \geq \ (d-1) \int_{\mathbb{S}^{d-1}} |u|^2 \, d\omega$$

Poincaré (Wirtinger) for 2D fermions: $u(\varphi + \pi) = -u(\varphi)$

$$\int_0^{2\pi} |u'|^2 \, d\varphi \ \geq \ \int_0^{2\pi} |u|^2 \, d\varphi$$

Statistical repulsion $\Leftarrow$ Poincaré inequality [2 anyons]

Poincaré for 2D fermions: $u(\varphi + \pi) = -u(\varphi)$

$$\int_0^\pi |u'|^2 \, d\varphi \;\geq\; \int_0^\pi |u|^2 \, d\varphi$$

Poincaré for anyons: $u(\varphi + \pi) = e^{i\pi\alpha} u(\varphi)$, $\alpha \in (-1, 1]$

$$\int_0^\pi |u'|^2 \, d\varphi \;\geq\; \alpha^2 \int_0^\pi |u|^2 \, d\varphi$$

Statistical repulsion $\Leftarrow$ Poincaré inequality [2 anyons]

Poincaré for 2D fermions: $u(\varphi+\pi) = -u(\varphi)$

$$\int_0^\pi |u'|^2\, d\varphi \;\geq\; \int_0^\pi |u|^2\, d\varphi$$

Poincaré for anyons: $u(\varphi+\pi) = e^{i\pi\alpha}u(\varphi)$, $\alpha \in (-1,1]$

$$\int_0^\pi |u'|^2\, d\varphi \;\geq\; \alpha^2 \int_0^\pi |u|^2\, d\varphi$$

If $\alpha \in \mathbb{R}$:

$$\int_0^\pi |u'|^2\, d\varphi \;\geq\; \min_{q\in\mathbb{Z}} |\alpha - 2q|^2 \int_0^\pi |u|^2\, d\varphi$$

$\Rightarrow$ **statistical repulsion** for a pair of anyons if $\alpha_2 > 0$

Statistical repulsion $\Leftarrow$ Poincaré inequality $[2+p$ anyons$]$

Poincaré for any number of 2D fermions: $u(\varphi+\pi) = -u(\varphi)$

$$\int_0^\pi |u'|^2 \, d\varphi \;\geq\; \int_0^\pi |u|^2 \, d\varphi$$

Poincaré for $2+p \leq N$ anyons: $u(\varphi+\pi) = e^{i\pi(2p+1)\alpha} u(\varphi)$,

$$\int_0^\pi |u'|^2 \, d\varphi \;\geq\; \min_{q\in\mathbb{Z}} |(2p+1)\alpha - 2q|^2 \int_0^\pi |u|^2 \, d\varphi$$

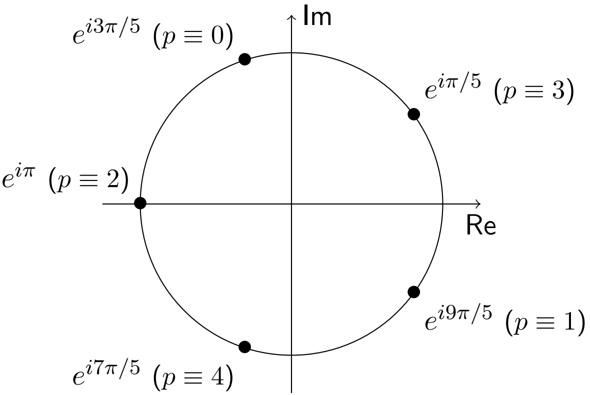
$\Rightarrow$ **statistical repulsion** (Hardy, extensivity, LT) for anyons if $0 <$

$$\alpha_{p\in N} := \mathrm{dist}\left(e^{i\pi(2p+1)\alpha}, 1\right), \qquad p = 0, 1, \ldots, N-2$$

By nearest neighbor and scaling, actually sufficient that $\alpha_2 > 0$

Abelian anyons, ex. $\alpha = 3/5$

$U_p = e^{(2p+1)i\pi\alpha}$



$\Rightarrow$ **Poincaré inequality** with $\alpha_{p\in N} = \begin{cases} 3/5, & p = 0, 4, 5, \ldots \\ 1/5, & p = 1, 3, \ldots \\ 1, & p = 2, 7, \ldots \end{cases}$

The **Thomas-Fermi** approximation for **fermions** in 2D:

$$E_N = \inf_{\Psi \in L^2_{\mathrm{asym}}} \langle \hat{T} + \hat{V} \rangle_\Psi \approx \inf_{\varrho \geq 0 : \int_{\mathbb{R}^2} \varrho = N} \int_{\mathbb{R}^2} \left[2\pi \varrho(\mathbf{x})^2 + V(\mathbf{x})\varrho(\mathbf{x}) \right] d\mathbf{x},$$

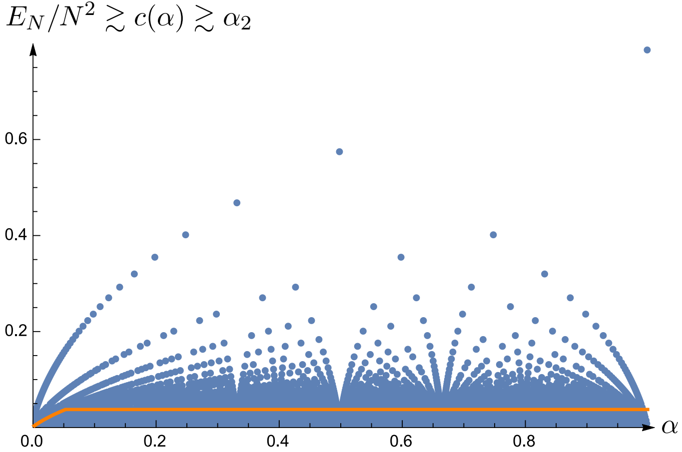
Theorem (**Lieb-Thirring** inequality): For any $\alpha \in \mathbb{R}$ and $\Psi \mapsto \varrho_\Psi$

$$\langle \hat{T}_\alpha \rangle_\Psi \gtrsim c(\alpha) \int_{\mathbb{R}^2} \varrho_\Psi(\mathbf{x})^2 \, d\mathbf{x}, \qquad c(\alpha) \sim \mathrm{dist}(\alpha, 2\mathbb{Z}) = \alpha_2.$$

Hence, to first order, the **degeneracy pressure** for the ideal anyon gas is governed by the 2-anyon simple exchange phase $e^{i\alpha\pi}$.

DL, Solovej '13,'14; Larson, DL 18'; DL, Seiringer '18; DL, Qvarfordt '20

Local exclusion $\Rightarrow$ degeneracy pressure for *ideal* anyon gas



Lieb-Thirring inequality: $\langle \Psi | \hat{H}_N | \Psi \rangle \gtrsim \int_{\mathbb{R}^2} \left(c(\alpha) \varrho_\Psi^2 + V \varrho_\Psi \right)$

DL, Solovej '13,'14; Larson, DL 18'; DL, Seiringer '18; DL, Qvarfordt '20

Lecture III: The almost-bosonic, extended, interacting anyon gas

Aim: basics of the average-field approach and CSS/afP model

Towards precise density functionals for anyons

Ground state approximation: $E_N/N \xrightarrow{N\to\infty}$ minimum of $\mathcal{E}[u]$:

The **Gross–Pitaevskii functional** for interacting **bosons**: $(g \in \mathbb{R})$

$$\mathcal{E}^{\mathrm{GP}}[u] := \int_{\mathbb{R}^2} \left(\left|(-i\nabla + \mathbf{A}_{\mathrm{ext}})u\right|^2 + V|u|^2 + g|u|^4 \right)$$

The **Thomas–Fermi functional** for **fermions**:

$$\mathcal{E}^{\mathrm{TF}}[u] := \int_{\mathbb{R}^2} \left(2\pi N|u|^4 + V|u|^2 \right)$$

An **"average-field" approximation** for **anyons**?

$$\mathcal{E}^{\mathrm{af}}[u] \approx \int_{\mathbb{R}^2} \left(2\pi\alpha N|u|^4 + V|u|^2 \right)$$

Mean-field ansatz: $\Psi(\mathrm{x}) = u(\mathbf{x}_1)u(\mathbf{x}_2)\ldots u(\mathbf{x}_N)$
$\Rightarrow$ Single particle $u \in L^2(\mathbb{R}^2)$ in magnetic field $2\pi\alpha N|u(\mathbf{x})|^2$

Approach: separate length scales $R \ll \ell \ll 1$

1. **Microscopic:** flux radius $R > 0$

2. **Mesoscopic:** typical interparticle scale $\ell \sim \varrho(\mathbf{x})^{-1/2}$

3. **Macroscopic:** scale of trap V (ex. 1 or $\omega^{-1/2}$)

Approach: separate length scales $R \ll \ell \ll 1$

1. **Microscopic:** flux radius $R > 0$, $B_j^R = 2\pi \sum_{k \neq j} \mathbb{1}_{B(\mathbf{x}_k,R)}/(\pi R^2)$
   N-anyon Hamiltonian, R-extended, spin-orbit coupling g:

$$H_N^{\alpha,R,g} := \sum_{j=1}^{N} \left[\left(-i\nabla_{\mathbf{x}_j} + \alpha \mathbf{A}_j^R\right)^2 + \frac{g}{2}\alpha B_j^R + V(\mathbf{x}_j) \right]$$

2. **Mesoscopic:** typical interparticle scale $\ell \sim \varrho(\mathbf{x})^{-1/2}$
   Chern–Simons–Schrödinger(–Ginzburg–Landau–Higgs)/
   "average-field-Pauli" functional at $\beta := (N-1)\alpha$, $\gamma = \ldots$?:

$$\mathcal{E}^{\mathrm{CSS}}_{\beta,\gamma,V}[u] := \int_{\mathbb{R}^2} \left(\left|(-i\nabla + \beta \mathbf{A}[|u|^2])u\right|^2 + \gamma |u|^4 + V|u|^2 \right)$$

3. **Macroscopic:** scale of trap V (ex. 1 or $\omega^{-1/2}$)
   Thomas–Fermi-type functional at $\beta \gg 1$ and $\gamma \propto \beta$:

$$\mathcal{E}^{\mathrm{TF}}_{G,V}[\varrho] := \int_{\mathbb{R}^2} \left(G\varrho^2 + V\varrho \right), \qquad G \approx 2\pi\beta c(\gamma/\beta)?$$

Almost-bosonic & self-interacting anyons $\Rightarrow$ DFT

Average-field ansatz close to bosons (identically distributed/BEC):

$$\Psi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N) \approx u(\mathbf{x}_1)u(\mathbf{x}_2)\ldots u(\mathbf{x}_N)$$

Statistics modelled by magnetic flux attachment to particles:

$$\operatorname{curl} \alpha \mathbf{A}_j(\mathbf{x}_j) = 2\pi\alpha \sum_{k \neq j} \delta_{\mathbf{x}_k}(\mathbf{x}_j) \quad \approx \quad 2\pi\alpha(N-1)|u(\mathbf{x}_j)|^2$$

Almost-bosonic anyons: finite total flux $\approx 2\pi\alpha N \to 2\pi\beta \in \mathbb{R}$.
For hard-core/interacting anyons, we add a scalar pair interaction $\approx \gamma\delta_0$ and consider the **Chern–Simons–Ginzburg–Landau–Higgs**/ "average-field-Pauli" functional of $u\colon \mathbb{R}^2 \to \mathbb{C}$, $\varrho = |u|^2$, $\int_{\mathbb{R}^2} \varrho = 1$:

$$\boxed{\mathcal{E}_{\beta,\gamma,V}[u] := \int_{\mathbb{R}^2} \left(\left|(\nabla + i\beta\mathbf{A}[|u|^2])u\right|^2 + \gamma|u|^4 + V|u|^2 \right)}$$

$$\mathbf{A}[\varrho] := (\nabla^\perp \log|\cdot|) * \varrho \quad \Rightarrow \quad \operatorname{curl} \beta\mathbf{A}[\varrho] = 2\pi\beta\varrho$$

Special coupling (Pauli / Bogomolnyi / self-dual): $\gamma = \pm 2\pi\beta$

Jackiw, Pi '90; Dunne '95,'98; ... DL, Rougerie '15; Correggi et al. '17-'19; Ataei, DL, Nguyen '24; +Girardot '25

Average-field approximation

1. Regularize/extend the flux to microscopic disks of radius $R > 0 \Rightarrow R$-**extended anyons**

$$\mathbf{A}_j^R := \sum_{k \neq j} \frac{(\mathbf{x}_j - \mathbf{x}_k)^\perp}{|\mathbf{x}_j - \mathbf{x}_k|_R^2}, \quad B_j^R = 2\pi \sum_{k \neq j} \left(\frac{\mathbb{1}_{B(0,R)}}{\pi R^2} \right) (\mathbf{x}_j - \mathbf{x}_k)$$

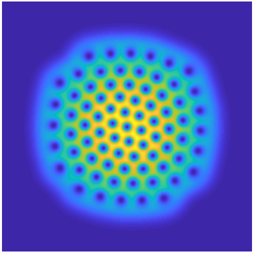
2. Reduce d.o.f. to a 1-body collective / mesoscopic average

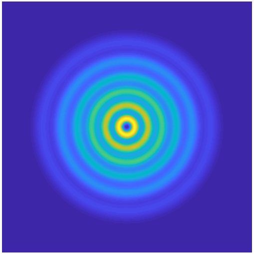
$$\mathbf{A}_j^R \approx \mathbf{A}^R[\varrho_\Psi] := \frac{\mathbf{x}^\perp}{|\mathbf{x}|_R^2} * \varrho_\Psi, \quad B_j^R \approx B^R[\varrho_\Psi]$$

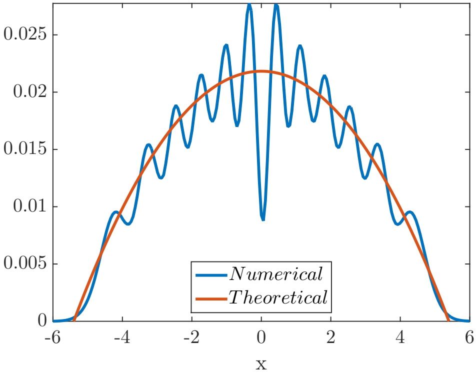
3. Take limit $R \to 0$ at a rate rel. to energy/scale of coll. problem $\Rightarrow$ self-consistent magnetic problem

Wilczek's book intro chapter '90; heuristics in Chen et al '89, Trugenberger '92
bosonic: DL, Rougerie '15; Girardot '20; Ataei, DL, Girardot '25
fermionic: Girardot, Rougerie '21; Levitt, DL, Rougerie '25

Almost-bosonic anyons: numerics







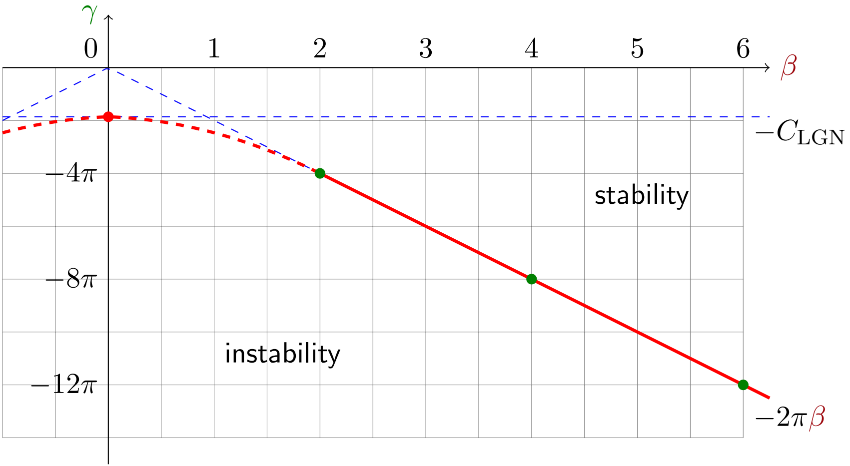
$\beta = 90$, $\gamma = 0$, $V(\mathbf{x}) = |\mathbf{x}|^2$

Averaged $|u_{\min}|^2$ and comparison with ϱ_{TF}

vortex lattice $\Rightarrow G \approx 2\pi\beta c$, $c = 2\sqrt{\pi}/3 \approx 1.18$

Correggi, Duboscq, DL, Rougerie '19

Magnetic stability: nonlinear Landau levels (NLLs)



$$\int_{\mathbb{R}^2} \left[|(\nabla + i\mathbf{A})u|^2 \pm B|u|^2 \right] = \int_{\mathbb{R}^2} \left| (\partial_1 \pm \mathrm{i}\partial_2)(e^{\pm\psi/2}u) \right|^2 e^{\mp\psi}, \text{ if } \mathbf{A} = -\tfrac{1}{2}\nabla^\perp \psi$$

Magnetic stability: $\mathcal{E}_{\beta,\gamma,0} \geq 0$

$u\colon \mathbb{R}^2 \to \mathbb{C}$ is a global section of a $U(1)$ complex line bundle over $\mathbb{R}^2 \cong \mathbb{C}$ with self-generated curvature $\mathbf{B} = 2\pi\beta|u|^2$. Min. KE/TF:

$$\gamma_*(\beta) := \inf\left\{ \frac{\mathcal{E}_{\beta,0,0}[u]}{\int_{\mathbb{R}^2} |u|^4} : u \in H^1(\mathbb{R}^2;\mathbb{C}), \int_{\mathbb{R}^2} |u|^2 = 1 \right\}$$

- **Diamagnetic inequality**:

$$\mathcal{E}_{\beta,0,0}[u] \geq \int_{\mathbb{R}^2} \big|\nabla|u|\big|^2$$

$\Rightarrow\ \gamma_*(\beta) \geq \gamma_*(0) = C_{\mathrm{LGN}} \approx 0.931 \times 2\pi$, optimal $H^1 \hookrightarrow L^4$ embedding constant of Ladyzhenskaya-Gagliardo-Nirenberg:

$$\int_{\mathbb{R}^2} |\nabla u|^2 \int_{\mathbb{R}^2} |u|^2 \geq C_{\mathrm{LGN}} \int_{\mathbb{R}^2} |u|^4$$

- **Pauli/SUSY/Bogomolnyi bound**: $\quad \Rightarrow\ \gamma_*(\beta) \geq 2\pi|\beta|$

$$\int_{\mathbb{R}^2} \big|(\nabla + i\mathbf{A})u\big|^2 \geq \pm \int_{\mathbb{R}^2} \mathbf{B}|u|^2, \qquad \mathbf{B} = \operatorname{curl} \mathbf{A}$$

Magnetic stability: $\mathcal{E}_{\beta,\gamma,0} \geq 0$

$$-\gamma \;\leq\; \gamma_*(\beta) := \inf\left\{ \frac{\mathcal{E}_{\beta,0,0}[u]}{\int_{\mathbb{R}^2} |u|^4} : u \in H^1(\mathbb{R}^2;\mathbb{C}), \int_{\mathbb{R}^2} |u|^2 = 1 \right\}$$

Theorem (Ataei, L., Nguyen '24)

i) *The function $\beta \mapsto \gamma_*(\beta)$ is Lipschitz and satisfies*

$$\gamma_*(\beta) > \max\{\gamma_*(0), 2\pi\beta\} \qquad \text{for every } 0 < \beta < 2,$$
$$\gamma_*(\beta) = 2\pi\beta \qquad \text{for every } \beta \geq 2.$$

ii) *Any minimizer, if it exists, is smooth. For small enough $0 < \beta < 2$, there exists a minimizer. For $\beta \geq 2$, minimizers exist if and only if $\beta \in 2\mathbb{N}$, and are of the form*

$$u = u_{P,Q} := \sqrt{\frac{2}{\pi\beta}}\,\frac{\overline{P'Q - PQ'}}{|P|^2 + |Q|^2},$$

where P, Q are two coprime and linearly independent complex polynomials satisfying $\max(\deg(P), \deg(Q)) = \beta/2$.

iii) *Finally, $u_{P,Q} = u_{\tilde{P},\tilde{Q}}$ if and only if $(P,Q) = \Lambda(\tilde{P},\tilde{Q})$ for some constant $\Lambda \in \mathbb{R}^+ \times \mathrm{SU}(2)$.*

Jackiw, Pi '90; Hagen '91; Horvathy, Yera '98 / Aharonov, Casher '79; Hong et al., Jackiw et al., Kim et al. 90'

Proof via a generalized Liouville equation

NLL/SUSY: $u(\mathbf{x}) = (4\pi\beta)^{-1/2} e^{\psi(\mathbf{x})/2}\overline{f(z)} \quad \Rightarrow \quad 4\pi\beta\varrho = e^{\psi}|f|^2,$

$$-\Delta \log(\varrho) = 4\pi\beta\varrho - 4\pi \sum_{j=1}^{n} \delta_{z_j} \quad \Leftrightarrow \quad \boxed{-\Delta\psi = |f|^2 e^{\psi}}$$

z_j zeros of the Wronskian $P'Q - PQ'$, $\quad \beta/2 - 1 \le n \le \beta - 2$

Theorem (Ataei, L., Nguyen '24)

Let $f\colon \mathbb{C} \to \mathbb{C}$ be a nonzero polynomial. All the weak solutions $\psi \in L^1_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R})$ such that $\int_{\mathbb{R}^2} |f|^2 e^{\psi} < \infty$ are of the form

$$\psi = \psi_{P,Q} := \log(8) - 2\log(|P|^2 + |Q|^2),$$

where P, Q are two coprime complex polynomials which satisfy $f = P'Q - PQ'$. Moreover, $\max(\deg(P), \deg(Q)) = \frac{\int_{\mathbb{R}^2} |f|^2 e^{\psi}}{8\pi}$, and $\psi_{P,Q} = \psi_{\tilde{P},\tilde{Q}}$ for some pairs of polynomials (P,Q) and $(\tilde{P},\tilde{Q})$ if and only if $(\tilde{P},\tilde{Q}) = \Lambda(P,Q)$, for some constant $\Lambda \in \mathrm{U}(2)$.

Liouville 1853; Jackiw, Pi 1990; Chen, Li '91,'93; Chou, Wan '94; Prajapat, Tarantello '01; Brito et al. '03,'05 ...

Radially symmetric solutions

- $\beta = 0$: "Townes' soliton": $u \propto \tau$ unique solution to the critical self-focusing 2D nonlinear Schrödinger equation

$$-\Delta u - |u|^2 u = -u$$

at $\gamma_*(0) = C_{\mathrm{LGN}} = \|\tau\|_{L^2}^2 /2 \approx 0.931 \times 2\pi$:

$$\int_{\mathbb{R}^2} |\nabla \tau|^2 \int_{\mathbb{R}^2} |\tau|^2 = C_{\mathrm{LGN}} \int_{\mathbb{R}^2} |\tau|^4$$

- $\beta = 2$: "*versiera*" / "the witch of Agnesi":

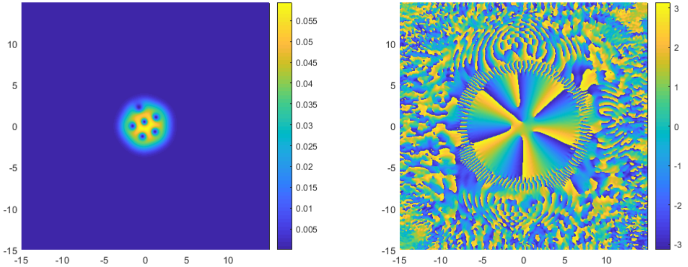
$$u \sim \frac{1}{\sqrt{\pi}} \frac{1}{|z|^2 + 1}$$

- $\beta = 2n$, $n > 1$: "vortex ring":

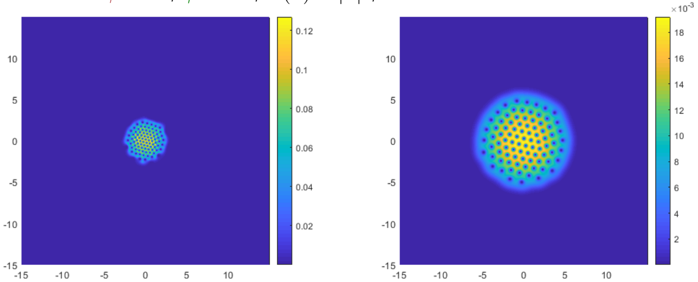
$$u \sim \sqrt{\frac{n}{\pi}} \frac{\bar{z}^{n-1}}{|z|^{2n} + 1}$$

Chiao, Garmire, Townes '64; Weinstein '83; Jackiw, Pi '90

Non-radial solutions require solving an **inverse Wronskian problem**.

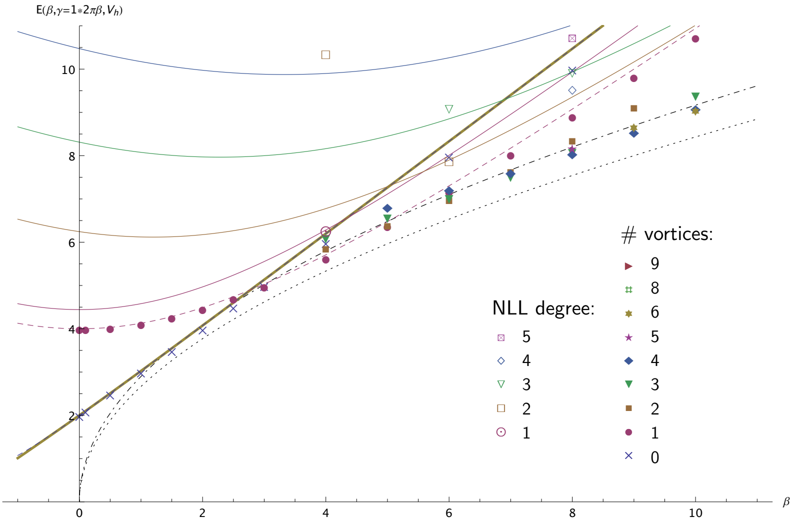


$\beta = 10$, $\gamma = 20\pi$, $V(\mathbf{x}) = |\mathbf{x}|^2$, $E \approx 9.066$



$\beta = 100$, $\gamma = -186\pi$, $E \approx 5.589$, resp. $\gamma = 200\pi$, $E \approx 28.620$

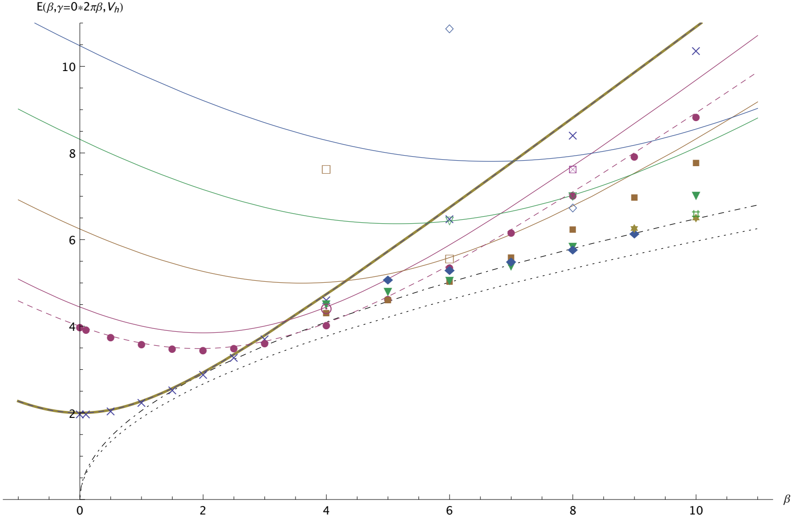
Ataei, Ellingsen, Getzner, Girardot, L, Nguyen '25



$$\beta \in [-1, 11], \quad \gamma = +2\pi\beta, \quad V(x,y) = x^2 + y^2, \quad G \approx 4\pi\beta c$$

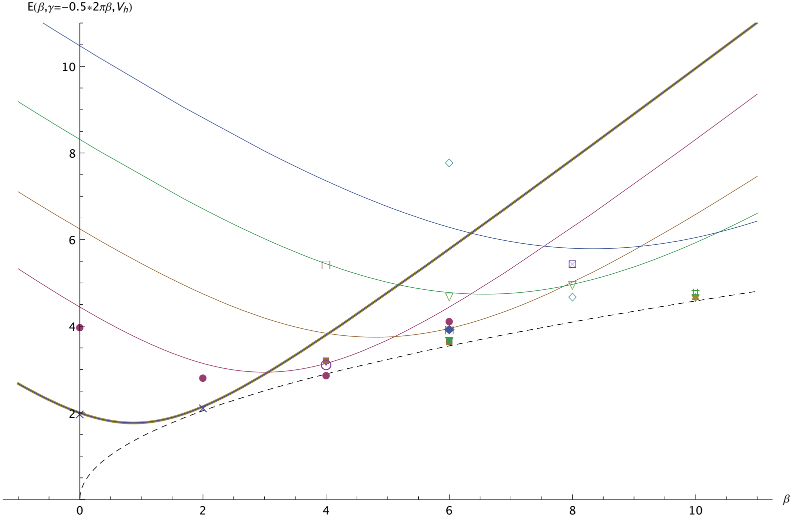
Ataei, Ellingsen, Getzner, Girardot, L, Nguyen '25

Macroscopics: energies $\beta \mapsto E^{\mathrm{CSS}}_{\beta,\gamma,V}$ (upper bounds)



$\beta \in [-1, 11], \qquad \gamma = 0, \qquad V(x,y) = x^2 + y^2, \qquad G \approx 2\pi\beta c$

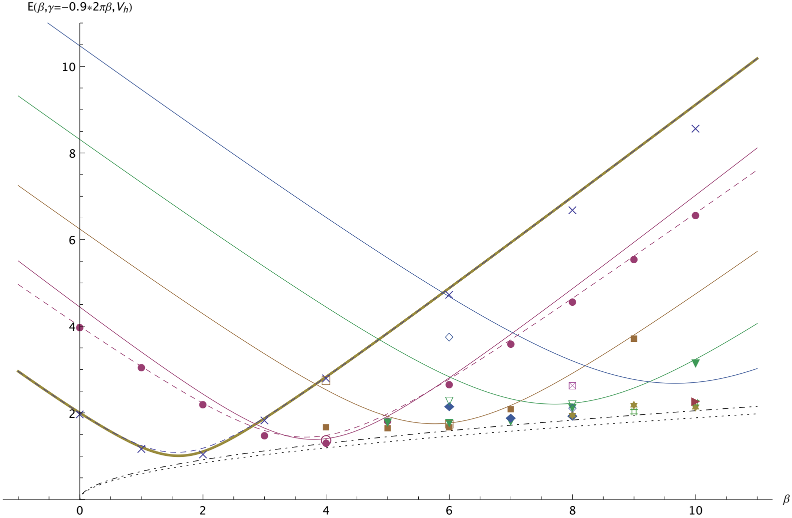
Correggi, Duboscq, L, Rougerie '19; Ataei, Ellingsen, Getzner, Girardot, L, Nguyen '25



$\beta \in [-1, 11], \quad \gamma = -\pi\beta, \quad V(x,y) = x^2 + y^2, \quad G \approx \pi\beta c$

Ataei, Ellingsen, Getzner, Girardot, L, Nguyen '25

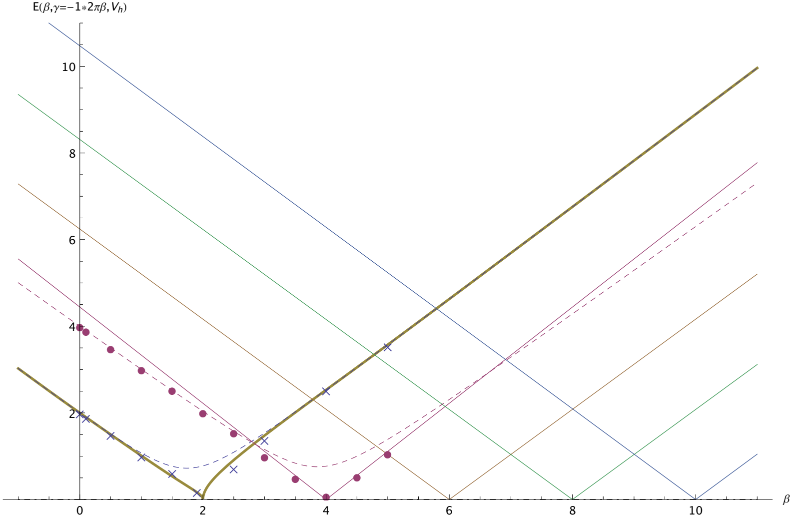
Macroscopics: energies $\beta \mapsto E^{\mathrm{CSS}}_{\beta,\gamma,V}$ (upper bounds)



$$\beta \in [-1, 11], \quad \gamma = -\tfrac{9}{5}\pi\beta, \quad V(x,y) = x^2 + y^2, \quad G \approx 0.2\pi\beta c$$

Ataei, Ellingsen, Getzner, Girardot, L, Nguyen '25

Macroscopics: energies $\beta \mapsto E^{\mathrm{CSS}}_{\beta,\gamma,V}$ (upper bounds)



$\beta \in [-1, 11], \quad \gamma = -2\pi\beta, \quad V(x,y) = x^2 + y^2, \quad G \approx 0?$

Ataei, Ellingsen, Getzner, Girardot, L, Nguyen '25

Lecture IV

Lecture IV: Nonabelian anyons & topological quantum computing

Aim: basics of nonabelian reps, anyons models & TQC

Application: Topological quantum computing

PRL **103,** 160501 (2009) PHYSICAL REVIEW LETT

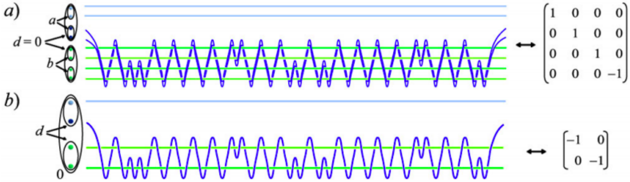


FIG. 2 (color online). "Effective qubit" gate construction for $\mathfrak{su}(2)_3$ anyons. Part (a) shows a braid in which a pair of anyons from the control qubit (blue) weaves around pairs of anyons in the target qubit (green). When either qubit is in the state $|0\rangle$, this braid produces the identity operation. When both control and target qubits are in the state $|1\rangle$, the braid consists of weaving a

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[Hormozi, Bonesteel, Simon]

History: Goldin, Menikoff & Sharp

VOLUME 54, NUMBER 6 PHYSICAL REVIEW LETTERS 11 FEBRUARY 1985

Comments on "General Theory for Quantum Statistics in Two Dimensions"

In a recent Letter[1] Wu provides, among other results, a derivation of exotic quantum statistics in two-dimensional space using Feynman path integrals, extending an argument given by Laidlaw and De Witt[2] for the three-dimensional case. The existence of such statistics and their physical interpretation in terms of local observables was previously obtained from the standpoint of group representations.[3] However, from this standpoint more general quantum theories than those in Ref. 1 are recognized to occur. Within a single framework, one obtains in addition the theories of particles obeying parastatistics, as well as particles with spin.[4,5] The description of these theories requires extension of the conventional path-integral formalism.

It has been shown[6] that quantum mechanics in R^s can be described by unitary representations of $\mathrm{Diff}(R^s)$, the group of diffeomorphisms of R^s which become trivial at infinity. Quantum statistics arises from certain *induced* representations of this group. $\mathrm{Diff}(R^s)$ acts in a natural way on n-particle configuration space Δ, whose fundamental group $\pi_1(\Delta)$ is the braid group B_n for $s=2$ and the symmetric group S_n for $s>2$. Then $\pi_1(\Delta)$ serves as a gauge group for the theory, and its unitary representations induce representations of $\mathrm{Diff}(R^s)$ describing the various particles statistics.

Wu states that "all possible quantum statistics in two-space are characterized by an angle parameter θ which interpolates between bosons and fermions." This assertion presupposes representations of the braid group which are one dimensional. There are, however, quantum theories obtained as representations of $\mathrm{Diff}(R^s)$ induced by higher-dimensional representations of $\pi_1(\Delta)$, corresponding to parastatistics (for S_n) or "unusual parastatistics" (for B_n). It does not seem widely recognized[7] that parastatistics[8] can also be described by Feynman path integrals on configuration space, taking the wave function ψ to be vector valued rather than scalar valued, and the propagator K to be an operator-valued function. Then the "weights" $\chi(\alpha)$ in Ref. 1, Eq. (1), for $\alpha \in \mathrm{B}_n$ or $\alpha \in \mathrm{S}_n$, can be unitary operators instead of phases, while $\int \exp(iS) D q$ remains a scalar quantity.

It should also be noted that even systems of *distinguishable* particles can be described by quantum theories with unusual phase shifts in two-dimensional space, because the *coordinate* space $\{(\mathbf{x}_1,\ldots,\mathbf{x}_N) \mid \mathbf{x}_i \in R^2, \mathbf{x}_i \neq \mathbf{x}_j \text{ for } i \neq j\}$ is not simply connected.[9] A possible example is that of quantized disturbances such as vortices in a thin film. Now *different* phase shifts can occur when different pairs of vortices circle each other by means of continuous paths in coordinate space. These phase shifts may be related to the relative vorticities. Thus it is not really the indistinguishability of the particles which accounts for the occurrence of unusual statistics in R^2, but the two-dimensionality of the space.

Quantum theories of particles with spin are also obtained as induced representations of $\mathrm{Diff}(R^2)$ or $\mathrm{Diff}(R^3)$. In one-particle configuration space, the gauge groups are the universal covering groups of the Lie groups $\mathrm{SL}(2,R)$ or $\mathrm{SL}(3,R)$ respectively.[5] Unitary representations of these groups can be decomposed with respect to the covering groups of $\mathrm{SO}(2)$ or $\mathrm{SO}(3)$. Now it is essential to consider the higher-dimensional representations in carrying out the inducing construction. In three dimensions, we obtain quantum theories of supermultiplets of particles with integer or half-integer spin, and in two dimensions particles with fractional spin, strictly from the representation theory of the diffeomorphism group. To express these systems in terms of path integrals, it appears necessary to enlarge the configuration space.[10]

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[6] The theoretical foundations for considering the group of diffeomorphisms of R^s in quantum mechanics are summarized in G. A. Goldin, R. Menikoff, and D. H. Sharp, Phys. Rev. Lett. **51**, 2246 (1983), and are also discussed in the papers in Ref. 5.

[7] See Ref. 2, as well as L. S. Schulman, *Techniques and Applications of Path Integration* (Wiley, New York, 1981).

[8] H. S. Green, Phys. Rev. **90**, 270 (1953); O. W. Greenberg, in *Mathematical Theory of Elementary Particles,* edited by R. Goodman and I. Segal (MIT Press, Cambridge, 1966), pp. 29–44.

[9] There is a natural homomorphism $h\colon \mathrm{B}_n \to \mathrm{S}_n$. The subgroup of B_n which maps to the identity in S_n is the fundamental group of this coordinate space.

[10] L. S. Schulman, Phys. Rev. **176**, 1558 (1968); J. F. Hamilton, Jr., and L. S. Schulman, J. Math. Phys. **12**, 160 (1971).

History: Fröhlich, Gabbiani, Kerler, Marchetti, ...

Proceedings of The Gibbs Symposium
Yale University, May 15–17, 1989

Quantum Statistics and Locality

JÜRG FRÖHLICH

CONTENTS

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Fault-tolerant quantum computation by anyons

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Received 20 May 2002

Abstract

A two-dimensional quantum system with anyonic excitations can be considered as a quantum computer. Unitary transformations can be performed by moving the excitations around each other. Measurements can be performed by joining excitations in pairs and observing the result of fusion. Such computation is fault-tolerant by its physical nature.
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1. Introduction

A quantum computer can provide fast solution for certain computational problems (e.g., factoring and discrete logarithm [1]) which require exponential time on an ordinary computer. Physical realization of a quantum computer is a big challenge for scientists. One important problem is decoherence and systematic errors in unitary transformations which occur in real quantum systems. From the purely theoretical point of view, this problem has been solved due to Shor's discovery of fault-tolerant quantum computation [2], with subsequent improvements [3–6]. An arbitrary quantum circuit can be simulated using imperfect gates, provided these gates are close to the ideal ones up to a constant precision δ. Unfortunately, the threshold value of δ is rather small;[1] it is very difficult to achieve this precision.

Needless to say, *classical* computation can be also performed fault-tolerantly. However, it is rarely done in practice because classical gates are reliable enough. Why is it possible? Let us try to understand the easiest thing—why classical

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E-mail addresses: kitaev@itp.ac.ru, kitaev@cs.caltech.edu.
[1] Actually, the threshold is not known. Estimates vary from 1/300 [7] to 10^{-6} [4].

Selected for a Viewpoint in *Physics*

PHYSICAL REVIEW B **83**, 075303 (2011)

Plasma analogy and non-Abelian statistics for Ising-type quantum Hall states

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We study the non-Abelian statistics of quasiparticles in the Ising-type quantum Hall states which are likely candidates to explain the observed Hall conductivity plateaus in the second Landau level, most notably the one at filling fraction $\nu = 5/2$. We complete the program started in V. Gurarie and C. Nayak, [Nucl. Phys. B **506**, 685 (1997)]. and show that the degenerate four-quasihole and six-quasihole wave functions of the Moore-Read Pfaffian state are orthogonal with equal constant norms in the basis given by conformal blocks in a $c = 1 + \frac{1}{2}$ conformal field theory. As a consequence, this proves that the non-Abelian statistics of the excitations in this state are given by the explicit analytic continuation of these wave functions. Our proof is based on a plasma analogy derived from the Coulomb gas construction of Ising model correlation functions involving both order and (at most two) disorder operators. We show how this computation also determines the non-Abelian statistics of collections of more than six quasiholes and give an explicit expression for the corresponding conformal block-derived wave functions for an arbitrary number of quasiholes. Our method also applies to the anti-Pfaffian wave function and to Bonderson-Slingerland hierarchy states constructed over the Moore-Read and anti-Pfaffian states.

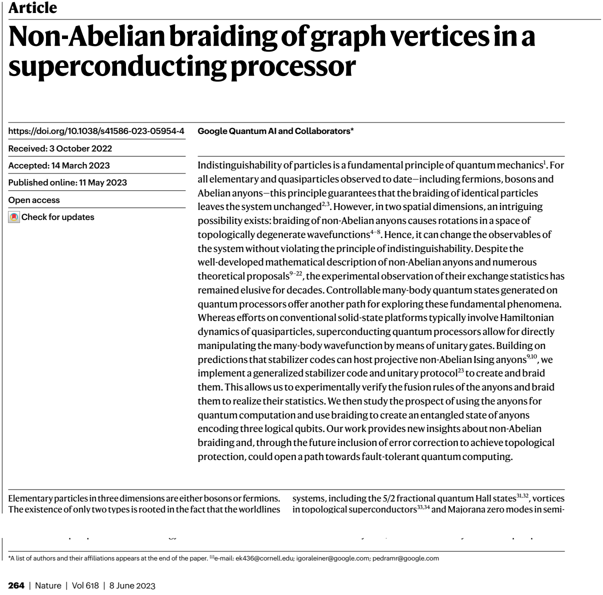
DOI: 10.1103/PhysRevB.83.075303 PACS number(s): 73.43.Lp, 05.30.Pr, 11.25.Hf

I. INTRODUCTION

Non-Abelian braiding statistics[1–7] is currently the subject of intense study, partly because the experimental observation of a non-Abelian anyon would be a remarkable milestone in fundamental science and partly because of its potential application to topologically fault-tolerant quantum information processing.[8–16] At present, the state which is the best candidate to support quasiparticles with non-Abelian braiding statistics is the experimentally observed $\nu = 5/2$ fractional quantum Hall state.[17–21] Efforts to observe non-Abelian anyons in this state[22–29] and harness them for quantum computation[23,30–33] are predicated entirely on two assumptions: (i) The observed state is in the same universality class as either the Moore-Read (MR) Pfaffian state[34] or the anti-Pfaffian state,[35,36] an assumption which is supported by numerical studies.[37–40] (There is another non-Abelian candidate, the so-called $SU(2)_2$ NAF state,[41] for this plateau, but it is not supported by numerics.) (ii) Quasiparticle excitations above these ground states are non-Abelian anyons. In order for this assumption to hold, it is necessary for there to be a degenerate set of n-quasiparticle states *and* for quasiparticle braiding to transform these states into each other in such a way that (2D) plasmas at a particular temperature and, therefore, is at the same level of rigor as the Berry's phase calculation for quasiparticles in the $\nu = 1/M$ Laughlin states.[55] In this paper, we supply such a proof by mapping matrix elements of the MR Pfaffian state to the partition function of a classical multi-component 2D plasma, possibly with magnetic charges. Our derivation extends and completes a partial result obtained in Ref. 45. Numerical studies provide very strong evidence that the plasmas corresponding to the $\nu = 1/M$ Laughlin states with $M \lesssim 70$ are in the screening phase.[56] Similar numerical evidence confirming that the plasma (described in our paper) corresponding to the $\nu = 1/2$ MR state is in the screening phase has recently also been obtained.[57]

One approach to the calculation of the braiding statistics of quasiparticles in fractional quantum Hall states is based on an idea due to Moore and Read.[34] These authors proposed to use the conformal blocks of conformal field theories[58,59] (CFTs) as trial wave functions for fractional quantum Hall effect states. The conformal blocks are the holomorphic parts of correlation functions. Unlike correlation functions, conformal blocks are not single valued. The conformal blocks which are used as trial wave functions for fractional quantum Hall effect states

History: Google et al.


Article

Non-Abelian braiding of graph vertices in a superconducting processor

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Google Quantum AI and Collaborators*

Indistinguishability of particles is a fundamental principle of quantum mechanics[1]. For all elementary and quasiparticles observed to date—including fermions, bosons and Abelian anyons—this principle guarantees that the braiding of identical particles leaves the system unchanged[2,3]. However, in two spatial dimensions, an intriguing possibility exists: braiding of non-Abelian anyons causes rotations in a space of topologically degenerate wavefunctions[4–8]. Hence, it can change the observables of the system without violating the principle of indistinguishability. Despite the well-developed mathematical description of non-Abelian anyons and numerous theoretical proposals[9–22], the experimental observation of their exchange statistics has remained elusive for decades. Controllable many-body quantum states generated on quantum processors offer another path for exploring these fundamental phenomena. Whereas efforts on conventional solid-state platforms typically involve Hamiltonian dynamics of quasiparticles, superconducting quantum processors allow for directly manipulating the many-body wavefunction by means of unitary gates. Building on predictions that stabilizer codes can host projective non-Abelian Ising anyons[9,10], we implement a generalized stabilizer code and unitary protocol[23] to create and braid them. This allows us to experimentally verify the fusion rules of the anyons and braid them to realize their statistics. We then study the prospect of using the anyons for quantum computation and use braiding to create an entangled state of anyons encoding three logical qubits. Our work provides new insights about non-Abelian braiding and, through the future inclusion of error correction to achieve topological protection, could open a path towards fault-tolerant quantum computing.

Elementary particles in three dimensions are either bosons or fermions. The existence of only two types is rooted in the fact that the worldlines systems, including the 5/2 fractional quantum Hall states[31,32], vortices in topological superconductors[33,34] and Majorana zero modes in semi-

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Geometric vs. magnetic anyon models

Lengthy discussion on mathematical definitions...

$$T = \sum_{j=1}^{N} \mathbf{p}_j^2$$

...$\Rightarrow$ $\hat{T}_\rho$ quantization of kinetic energy, labeled by $\rho\colon B_N \to \mathrm{U}(D)$

For abelian anyons, $\hat{T}_\rho$ is equivalent to the **magnetic** operator

$$\hat{T}_\alpha = \sum_{j=1}^{N} \left(-i\nabla_{\mathbf{x}_j} + \alpha \sum_{k \neq j} \frac{(\mathbf{x}_j - \mathbf{x}_k)^\perp}{|\mathbf{x}_j - \mathbf{x}_k|^2} \right)^2$$

acting on the **bosonic** Hilbert space $L^2_{\mathrm{sym}}(\mathbb{R}^{2N})$.

Abelian vs. non-abelian representations

An N-**anyon wave function** is *locally* a map $\Psi\colon \mathcal{C}^N \to \mathcal{F}$,

$\mathcal{F}$ Hilbert space of 'internal degrees of freedom' on which B_N acts:

$$\rho\colon B_N \to \mathrm{U}(\mathcal{F})$$

$$\langle \Phi, \Psi \rangle = \int_{\mathcal{C}^N} \langle \Phi(X), \Psi(X) \rangle_{\mathcal{F}} \, dX, \quad \|\Psi\|^2 = \int_{\mathcal{C}^N} |\Psi|^2_{\mathcal{F}} = 1$$

Irreducible abelian anyons: $\mathcal{F} = \mathbb{C}$,

$$\rho(\sigma_j) = e^{i\alpha\pi}$$

Reducible abelian anyons: $\mathcal{F} = \mathbb{C}^D$, $D > 1$,

$$\rho(\sigma_j) \sim \mathrm{diag}(e^{i\beta_1\pi}, \ldots, e^{i\beta_D\pi}) \qquad \forall j$$

Non-abelian anyons: $\mathcal{F} = \mathbb{C}^D$, $D > N - 3$ (if $N > 6$),

$$\rho(\sigma_j)\rho(\sigma_k) \neq \rho(\sigma_k)\rho(\sigma_j) \qquad \text{for some } j \neq k.$$

Free anyons: demand that *locally*, i.e. on any topologically *trivial* open subset $\Omega \subseteq \mathcal{C}^N$, the particles behave like usual free non-relativistic *distinguishable* particles (Schrödinger rep.)

$$\hat{T}_\Omega = \sum_{j=1}^{N} (-i\nabla_{\mathbf{x}_j})^2 \qquad \text{on} \qquad \Psi \in C_c^\infty(\Omega; \mathcal{F}) \subseteq L^2(\Omega; \mathcal{F}),$$

with some fiber (local/internal) Hilbert space $\mathcal{F} \cong \mathbb{C}^D$.

Fiber bundles: *globally* on $\mathcal{C}^N$ we should consider a hermitian vector bundle $E \to \mathcal{C}^N$ with fiber $\mathcal{F}$, endowed with a (locally) *flat* connection $\mathcal{A}$. **Wave functions** Ψ are L^2-sections of this bundle.

Theorem: There is a 1-to-1 correspondence between such flat bundles and representations $\rho\colon \pi_1(\mathcal{C}^N) = B_N \to \mathrm{U}(\mathcal{F})$.

Definition: A **geometric N-anyon model** is such a rep. $\rho \Rightarrow \hat{T}_\rho$

Geometric anyon models: alt. definition

Consider the **covering space**, i.e. the space of paths from a fixed base point modulo homotopy equivalences,

$$\tilde{\mathcal{C}}^N \to \mathcal{C}^N \qquad \text{with fiber } B_N.$$

An **N-anyon wave function** $\Psi \in L^2_\rho$ is a ρ-equivariant function

$$\Psi \colon \tilde{\mathcal{C}}^N \to \mathcal{F}, \quad \Psi(\gamma.\tilde{X}) = \rho([\gamma])\Psi(\tilde{X}), \quad \gamma \text{ loop in } \mathcal{C}^N,$$

$$\langle \Phi, \Psi \rangle_{L^2_\rho} := \int_{\mathcal{C}^N} \langle \Phi(\tilde{X}), \Psi(\tilde{X}) \rangle_{\mathcal{F}} \, dX.$$

The **Sobolev space** H^1_ρ is the closure of smooth ρ-equivariant functions $\Psi \colon \tilde{\mathcal{C}}^N \to \mathcal{F}$, with the projection of $\operatorname{supp} \Psi$ to $\mathcal{C}^N$ compact, w.r.t.

$$\langle \Phi, \Psi \rangle_{H^1_\rho} := \int_{\mathcal{C}^N} \Big(\langle \Phi(\tilde{X}), \Psi(\tilde{X}) \rangle_{\mathcal{F}} + \langle \nabla \Phi(\tilde{X}), \nabla \Psi(\tilde{X}) \rangle_{\mathcal{F}^{2N}} \Big) \, dX.$$

The associated operator is $\hat{T}_\rho \geq 0$ (Friedrichs extension)

Magnetic anyon models & transmutability

Sections Ψ are ρ-equivariant functions $\Psi_\rho \colon \tilde{\mathcal{C}}^N \to \mathcal{F}$,

$$\Psi_\rho(\gamma.\tilde{X}) = \rho([\gamma])\Psi_\rho(\tilde{X}), \quad \gamma \text{ loop in } \mathcal{C}^N.$$

Local **gauge transformation** $\Psi \to u\Psi$ where $u \colon \Omega \to \mathrm{U}(\mathcal{F})$:

$$\hat{T}_\Omega = -(\nabla + \mathcal{A})^2, \qquad \mathcal{A}(\tilde{X}) := u(\tilde{X})^{-1}\nabla u(\tilde{X})$$

If $u_\rho \colon \tilde{\mathcal{C}}^N \to \mathrm{U}(\mathcal{F})$ is a *global* section of the associated principal bundle,

$$u_\rho(\gamma.\tilde{X}) = \rho([\gamma])u_\rho(\tilde{X}), \quad \gamma \text{ loop in } \mathcal{C}^N,$$

then we have a tranformation to *trivial* bundle $\Psi_1 \in L^2_{\mathrm{sym}}(\mathbb{R}^{2N};\mathcal{F})$:

$$\Psi_\rho = u_\rho\Psi_1 \quad \Leftrightarrow \quad \Psi_1 = u_\rho^{-1}\Psi_\rho$$

Definition: A **transmutable N-anyon model** is an N-anyon model $\rho \colon B_N \to \mathrm{U}(\mathcal{F})$ such that its corresponding flat principal bundle $P \to \mathcal{C}^N$ is topologically trivial.

So, *transmutable* models ρ may equivalently be described using **bosons** (or fermions) with gauge potentials $\mathcal{A}\colon \mathbb{R}^{2N} \setminus \triangle \to \mathfrak{u}(\mathcal{F})$.

Obstacle: Only some of rep. theory of B_N and topology known.

Theorem: Any *abelian* model is transmutable. [Dowker '85, Mund, Schrader '95]

Theorem: Any rep $\rho\colon B_N \to \mathrm{U}(D)$, $N > 6$, is abelian if $D < N - 2$. [Formanek '96]

- In the non-abelian case typically $D \sim c^N$ for some $c > 0$.
- NACS: $\rho \sim \rho_1^{\otimes N}$ is transmutable.
- Transmutability of a bundle $E \to \mathcal{C}^N$ improves with $E \oplus E$...

Exchange vs. exclusion: anyons

Take a **local approach** to exchange and exclusion. [DL, Solovej '13]

Statistical repulsion manifests in three ways (at least):

1. effective *scalar* pairwise repulsion $\Rightarrow \Psi \to 0$ at $\triangle$
2. local exclusion principle: $E_N \geq \pi^2(N-1)_+$
3. degeneracy pressure, ex. Thomas-Fermi or Lieb-Thirring

Given $\rho = \rho_N$, consider '**exchange operator**' ($2p+1$ braidings):

$$U_p := \rho(\sigma_1\sigma_2\ldots\sigma_p\sigma_{p+1}\sigma_p\ldots\sigma_2\sigma_1), \quad p \in \{0,1,\ldots,N-2\}$$

and '**exchange parameters**' for p enclosed or n involved particles

$$\beta_p := \min\{\beta \in [0,1] : e^{i\beta\pi} \text{ or } e^{-i\beta\pi} \text{ is an eigenvalue of } U_p\}$$

$$\alpha_n := \min_{p\in\{0,1,2,\ldots,n-2\}} \beta_p, \quad n \in \{0,1,\ldots,N\}.$$

Statistical repulsion $\Leftarrow$ Poincaré inequality

Hardy inequality for fermions in $\mathbb{R}^d$: [Hoffmann-Ostenhof^2, Laptev, Tidblom '08]

$$\hat{T}_{\rho=\mathrm{sign}} \geq \frac{d^2}{N} \sum_{1 \leq j < k \leq N} \frac{1}{|\mathbf{x}_j - \mathbf{x}_k|^2}$$

Poincaré for fermions: $u(-\omega) = -u(\omega)$, $\omega \in \mathbb{S}^{d-1}$ relative angles

$$\int_{\mathbb{S}^{d-1}} |\nabla_\omega u|^2 \, d\omega \;\geq\; (d-1) \int_{\mathbb{S}^{d-1}} |u|^2 \, d\omega$$

Poincaré for 2D fermions: $u(\varphi + \pi) = -u(\varphi)$

$$\int_0^{2\pi} |u'|^2 \, d\varphi \;\geq\; \int_0^{2\pi} |u|^2 \, d\varphi$$

Statistical repulsion $\Leftarrow$ Poincaré inequality [2 anyons]

Poincaré for 2D fermions: $u(\varphi+\pi) = -u(\varphi)$

$$\int_0^\pi |u'|^2 \, d\varphi \;\geq\; \int_0^\pi |u|^2 \, d\varphi$$

Poincaré for **abelian anyons**: $u(\varphi+\pi) = e^{i\pi\alpha}u(\varphi)$, $\alpha \in (-1,1]$

$$\int_0^\pi |u'|^2 \, d\varphi \;\geq\; \alpha^2 \int_0^\pi |u|^2 \, d\varphi$$

Poincaré for **non-abelian anyons**: $u(\varphi+\pi) = U_0 u(\varphi)$, $U_0 \in \mathrm{U}(\mathcal{F})$

$$\int_0^\pi |u'|^2 \, d\varphi \;\geq\; \beta_0^2 \int_0^\pi |u|^2 \, d\varphi$$

$$\beta_0 := \min\{\beta \in [0,1] : e^{i\beta\pi} \text{ or } e^{-i\beta\pi} \text{ is an eigenvalue of } U_0\}$$

$\Rightarrow$ **statistical repulsion** for a pair of anyons if $\beta_0 > 0$

Statistical repulsion $\Leftarrow$ Poincaré inequality $[2+p$ anyons$]$

Poincaré for 2D fermions: $u(\varphi+\pi) = -u(\varphi)$

$$\int_0^\pi |u'|^2 \, d\varphi \;\geq\; \int_0^\pi |u|^2 \, d\varphi$$

Poincaré for **abelian anyons**: $u(\varphi+\pi) = e^{i\pi(2p+1)\alpha} u(\varphi)$,

$$\int_0^\pi |u'|^2 \, d\varphi \;\geq\; \min_{q\in\mathbb{Z}} |(2p+1)\alpha - 2q|^2 \int_0^\pi |u|^2 \, d\varphi$$

Poincaré for **non-abelian anyons**: $u(\varphi+\pi) = U_p u(\varphi)$, $U_p \in \mathrm{U}(\mathcal{F})$

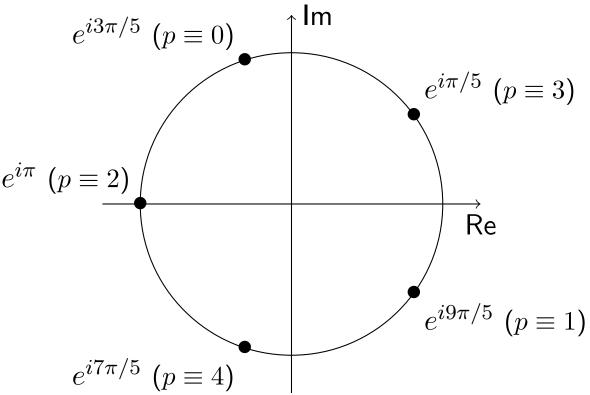
$$\int_0^\pi |u'|^2 \, d\varphi \;\geq\; \beta_p^2 \int_0^\pi |u|^2 \, d\varphi$$

$\beta_p := \min\{\beta \in [0,1] : e^{i\beta\pi}$ or $e^{-i\beta\pi}$ is an eigenvalue of $U_p\}$

$\Rightarrow$ **statistical repulsion** (Hardy, extensivity, LT) for anyons if $\beta_p > 0$

Abelian anyons, ex. $\alpha = 3/5$

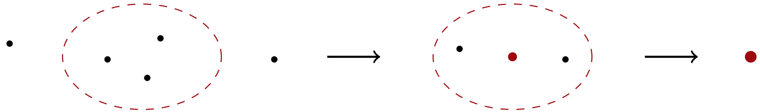
$U_p = e^{(2p+1)i\pi\alpha}$



$\Rightarrow$ **Poincaré inequality** with $\beta_p = \begin{cases} 3/5, & p = 0, 4, 5, \ldots \\ 1/5, & p = 1, 3, \ldots \\ 1, & p = 2, 7, \ldots \end{cases}$

Algebraic anyon models

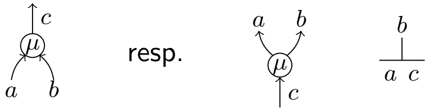
Braided fusion categories... Idea: **zoom out**



Labels / topological charges / particle types:

$$\mathcal{L} = \{a, b, c, \ldots\} = \{1, a, \bar{a}, b, \bar{b}, \ldots\}$$

Fusion / splitting diagrams:



Span spaces $V_{ab}^{c} \cong V_{c}^{ab}$ of dimension N_{ab}^{c} = number of ways fusion/splitting can occur.

Algebraic anyon models: Fusion

Fusion algebra: $a, b \in \mathcal{L}$,

$$a \times b = \sum_{c \in \mathcal{L}} N_{ab}^{c}\, c$$

The model turns out to be abelian if there is a unique result of fusion,

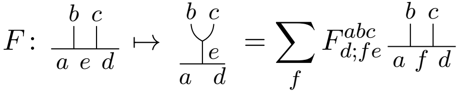
$$a \times b = c$$

Typically, $N_{ab}^{c} \in \{0, 1\}$, i.e. *multiplicity-free* models.

We write $c \in a \times b$ if $N_{ab}^{c} \neq 0$. Sums only over allowed indices.

Associativity of fusion: $(a \times b) \times c = a \times (b \times c)$

$\Rightarrow$ **F operator** (isomorphism on 2-split diagrams):

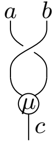


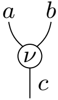
Algebraic anyon models: Braiding

Commmutativity of fusion: $a \times b = b \times a$

$\Rightarrow$ **R operator** (isomorphism on 1-split diagrams): $R_c^{ab} \in \mathrm{U}(V_c^{ab})$

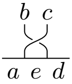
$R^{ab}:$ 

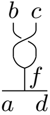
$\mapsto$ 

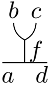
$= \sum_\nu \left[R_c^{ab}\right]_{\nu\mu}$ 

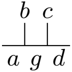
.

$\Rightarrow$ **B operator** (isomorphism on 2-split diagrams): $B := FRF^{-1}$

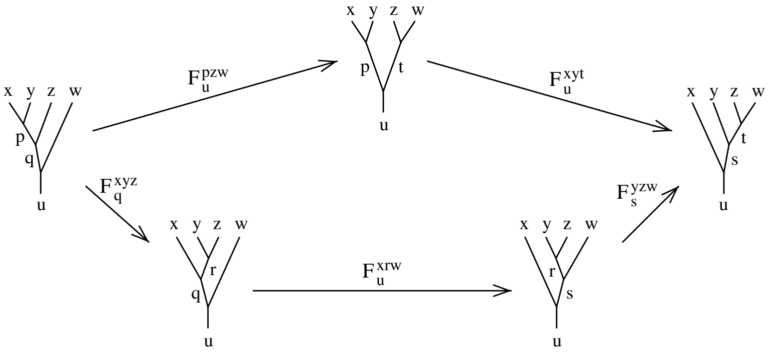


$= \sum_f \left(F^{-1}\right)^{acb}_{d;fe}$ 

$= \sum_f R_f^{bc} \left(F^{-1}\right)^{acb}_{d;fe}$ 

$= \sum_g \sum_f F^{abc}_{d;gf} R_f^{bc} \left(F^{-1}\right)^{acb}_{d;fe}$ 

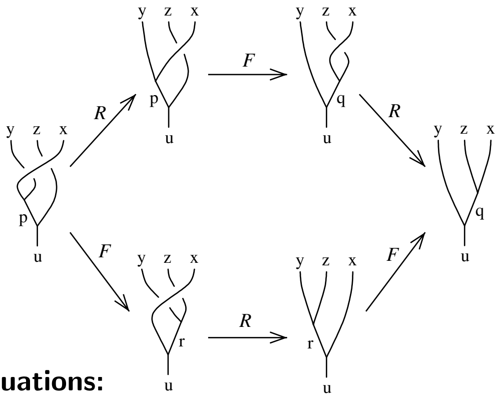
Algebraic anyon models: Consistency conditions



Pentagon equation:

$$F^{aby}_{e;xu} F^{ucd}_{e;yv} = \sum_{w \in \mathcal{L}} F^{abc}_{v;wu} F^{awd}_{e;xv} F^{bcd}_{x;yw},$$

Algebraic anyon models: Consistency conditions



Hexagon equations:

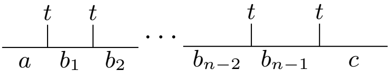
$$R_p^{yx} F_{u;qp}^{yxz} R_q^{zx} = \sum_{r \in \mathcal{L}} F_{u;rp}^{xyz} R_u^{rx} F_{u;qr}^{yzx},$$

$$\left(R_p^{yx}\right)^{-1} F_{u;qp}^{yxz} \left(R_q^{zx}\right)^{-1} = \sum_{r \in \mathcal{L}} F_{u;rp}^{xyz} \left(R_u^{rx}\right)^{-1} F_{u;qr}^{yzx},$$

Algebraic anyon models: Exchange operators

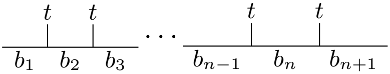
Standard splitting spaces:

$V_c^{a,t^n} = \mathrm{Span}\Big\{$



$:$ for all possible $b_j \Big\}$

$V_*^{*,t^n} = \mathrm{Span}\Big\{$

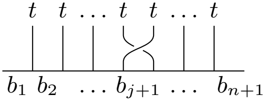


$:$ for all possible $b_j \Big\}$

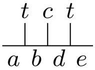
$$V_*^{*,t^n} = \bigoplus_{\substack{a\in\mathcal{L} \\ c\in a\times t^n}} V_c^{a,t^n}$$

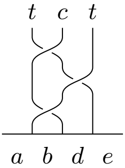
Defines a representation $\rho_n\colon B_n \to \mathrm{U}(V_*^{*,t^n})$:

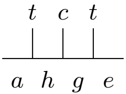
$\rho_n(\sigma_j)$:

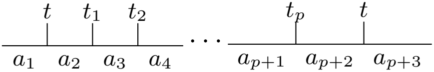


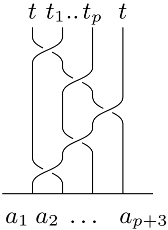
Algebraic anyon models: Exchange operators

$U_{t,c,t}:$ 

$\mapsto$ 

$= \sum_{f,g,h} B^{act}_{d;fb}\, B^{ftt}_{e;gd}\, B^{atc}_{g;hf}$ 

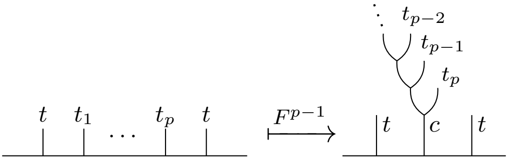
$U_{t,\{t_1,\ldots,t_p\},t}:$ 

$\mapsto$ 

If the anyon type $t = t_1 = \ldots = t_p$ is understood: $U_p := U_{t,t^p,t}$

$$U_p = \rho_n(\sigma_1\sigma_2\ldots\sigma_p\sigma_{p+1}\sigma_p\ldots\sigma_2\sigma_1)$$

Exchange in algebraic anyon models



Theorem ([DL, Qvarfordt])

The exchange operator of two t's around $t_1, \dots, t_p$ is given by

$$U_{t,\{t_1,\dots,t_p\},t} \sim \bigoplus_{c \in t_1 \times \dots \times t_p} U_{t,c,t}$$

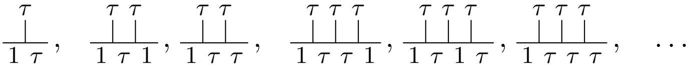
where c is a possible result of the fusion $t_1 \times t_2 \times \ldots \times t_p$, counted with multiplicity.

Fibonacci anyons

Model: $\mathcal{L} = \{1, \tau\}$,

$$\tau \times \tau = 1 + \tau$$

N-particle basis states:



$$\tau^N = \mathrm{Fib}(N-1)1 + \mathrm{Fib}(N)\tau,$$

where $\mathrm{Fib}(0) = 0$, $\mathrm{Fib}(1) = 1$, $\mathrm{Fib}(n) = \mathrm{Fib}(n-2) + \mathrm{Fib}(n-1)$.

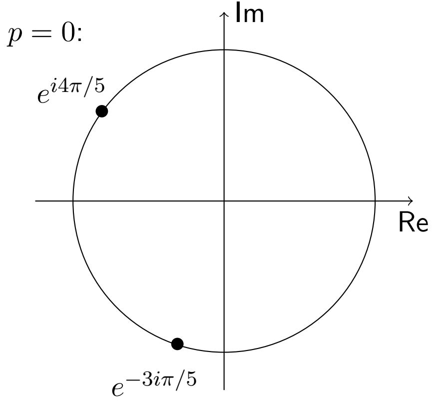
$$F_1^{\tau\tau\tau} = 1 \qquad \text{and} \qquad F_\tau^{\tau\tau\tau} = \begin{bmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi^{-1} \end{bmatrix}, \ \phi = \frac{1+\sqrt{5}}{2},$$

$$R_1^{\tau\tau} = e^{4\pi i/5}, \qquad R_\tau^{\tau\tau} = e^{-3\pi i/5}.$$

An **N-anyon Fibonacci model**: $\mathcal{F} = V_*^{1,\tau^N} \cong \mathbb{C}^D$, $D = \mathrm{Fib}(N+1)$

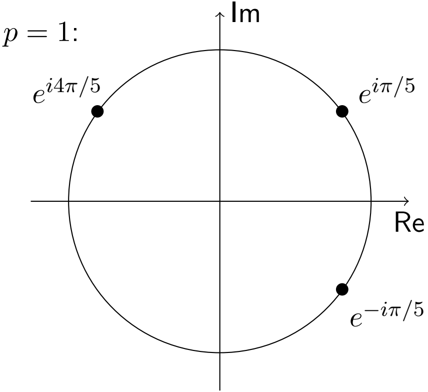
Fibonacci anyons: Exchange eigenvalues

$$U_{\tau,\tau^p,\tau} \sim U_{\tau,1,\tau}^{\oplus\,\mathrm{Fib}(p-1)} \oplus U_{\tau,\tau,\tau}^{\oplus\,\mathrm{Fib}(p)},$$
$$\mathrm{spec}(U_{\tau,1,\tau}) = \{e^{4\pi i/5}, e^{-3\pi i/5}\}, \;\; \mathrm{spec}(U_{\tau,\tau,\tau}) = \{e^{4\pi i/5}, e^{\pi i/5}, e^{-\pi i/5}\}$$



Fibonacci anyons: Exchange eigenvalues

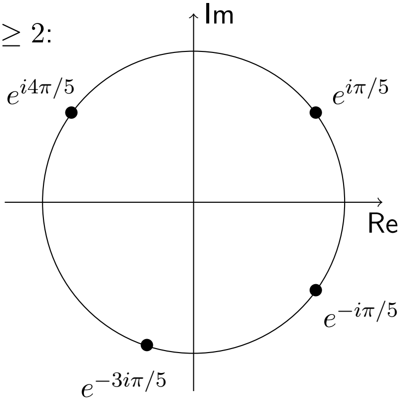
$$U_{\tau,\tau^p,\tau} \sim U_{\tau,1,\tau}^{\oplus\,\mathrm{Fib}(p-1)} \oplus U_{\tau,\tau,\tau}^{\oplus\,\mathrm{Fib}(p)},$$
$$\mathrm{spec}(U_{\tau,1,\tau}) = \{e^{4\pi i/5}, e^{-3\pi i/5}\}, \;\; \mathrm{spec}(U_{\tau,\tau,\tau}) = \{e^{4\pi i/5}, e^{\pi i/5}, e^{-\pi i/5}\}$$



Fibonacci anyons: Exchange eigenvalues

$$U_{\tau,\tau^p,\tau} \sim U_{\tau,1,\tau}^{\oplus\,\mathrm{Fib}(p-1)} \oplus U_{\tau,\tau,\tau}^{\oplus\,\mathrm{Fib}(p)},$$
$$\mathrm{spec}(U_{\tau,1,\tau}) = \{e^{4\pi i/5}, e^{-3\pi i/5}\}, \;\; \mathrm{spec}(U_{\tau,\tau,\tau}) = \{e^{4\pi i/5}, e^{\pi i/5}, e^{-\pi i/5}\}$$

$p \geq 2$:



$\Rightarrow$ **Poincaré inequality** with $\beta_0 = 3/5$ and $\beta_{p\geq 1} = 1/5$

Model: $\mathcal{L} = \{1, \psi, \sigma\}$,

$$\sigma \times \sigma = 1 + \psi, \qquad \sigma \times \psi = \sigma, \qquad \psi \times \psi = 1$$

$$\begin{array}{c}\sigma\\ \hline 1\ \sigma\end{array},\quad \begin{array}{c}\sigma\ \sigma\\ \hline 1\ \sigma\ 1\end{array},\ \begin{array}{c}\sigma\ \sigma\\ \hline 1\ \sigma\ \psi\end{array},\quad \begin{array}{c}\sigma\ \sigma\ \sigma\\ \hline 1\ \sigma\ 1\ \sigma\end{array},\ \begin{array}{c}\sigma\ \sigma\ \sigma\\ \hline 1\ \sigma\ \psi\ \sigma\end{array},\quad \begin{array}{c}\sigma\ \sigma\ \sigma\ \sigma\\ \hline 1\ \sigma\ 1\ \sigma\ 1\end{array},\ \begin{array}{c}\sigma\ \sigma\ \sigma\ \sigma\\ \hline 1\ \sigma\ \psi\ \sigma\ 1\end{array},\quad \ldots$$

$$\sigma^{2n+1} = 2^n \sigma, \qquad \sigma^{2n} = 2^{n-1}(1+\psi)$$

$$F_{\sigma}^{\sigma\sigma\sigma} = \pm \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad F_{\psi;\sigma\sigma}^{\sigma\psi\sigma} = F_{\sigma;\sigma\sigma}^{\psi\sigma\psi} = -1,$$

$$R_1^{\sigma\sigma} = e^{-\pi i/8}, \quad R_{\psi}^{\sigma\sigma} = e^{3\pi i/8}, \quad R_{\sigma}^{\sigma\psi} = R_{\sigma}^{\psi\sigma} = -i, \quad R_1^{\psi\psi} = -1.$$

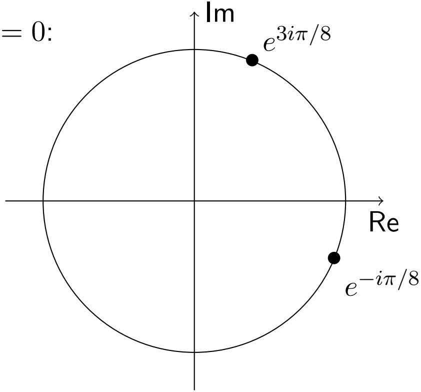
An N-**anyon Ising model**: $\hat{T}_{\rho_N}$ with $\mathcal{F} = V_*^{1,\sigma^N} \cong \mathbb{C}^D$, $D = 2^{\lfloor N/2 \rfloor}$

Ising anyons: Exchange eigenvalues

$$U_{\sigma,\sigma^{2n+1},\sigma} \sim U_{\sigma,\sigma,\sigma}^{\oplus 2^n}, \quad U_{\sigma,\sigma^{2n},\sigma} \sim U_{\sigma,1,\sigma}^{\oplus 2^{n-1}} \oplus U_{\sigma,\psi,\sigma}^{\oplus 2^{n-1}},$$

$\mathrm{spec}(U_{\sigma,\sigma,\sigma}) = \{e^{-\pi i/8}, e^{7\pi i/8}\}$, $\mathrm{spec}(U_{\sigma,1,\sigma}) = \{e^{-\pi i/8}, e^{3\pi i/8}\}$, $\mathrm{spec}(U_{\sigma,\psi,\sigma}) = \{e^{-5\pi i/8}, e^{7\pi i/8}\}$

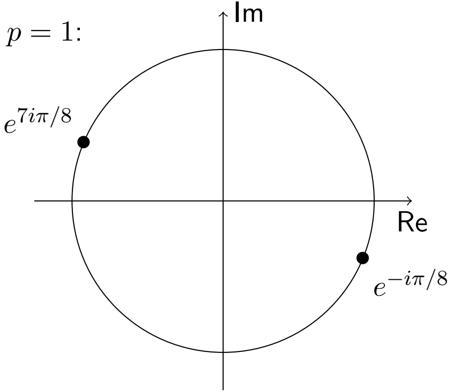
$p = 0$:



Ising anyons: Exchange eigenvalues

$$U_{\sigma,\sigma^{2n+1},\sigma} \sim U_{\sigma,\sigma,\sigma}^{\oplus 2^n}, \quad U_{\sigma,\sigma^{2n},\sigma} \sim U_{\sigma,1,\sigma}^{\oplus 2^{n-1}} \oplus U_{\sigma,\psi,\sigma}^{\oplus 2^{n-1}},$$

$\mathrm{spec}(U_{\sigma,\sigma,\sigma}) = \{e^{-\pi i/8}, e^{7\pi i/8}\}$, $\mathrm{spec}(U_{\sigma,1,\sigma}) = \{e^{-\pi i/8}, e^{3\pi i/8}\}$, $\mathrm{spec}(U_{\sigma,\psi,\sigma}) = \{e^{-5\pi i/8}, e^{7\pi i/8}\}$

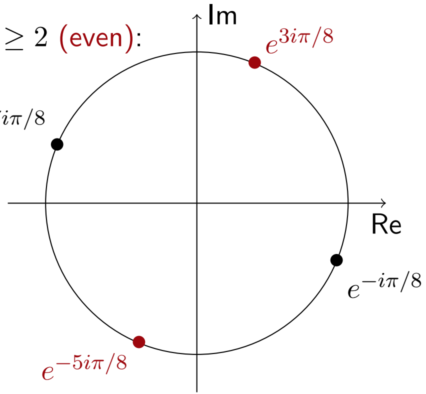


Ising anyons: Exchange eigenvalues

$$U_{\sigma,\sigma^{2n+1},\sigma} \sim U_{\sigma,\sigma,\sigma}^{\oplus 2^n}, \quad U_{\sigma,\sigma^{2n},\sigma} \sim U_{\sigma,1,\sigma}^{\oplus 2^{n-1}} \oplus U_{\sigma,\psi,\sigma}^{\oplus 2^{n-1}},$$

$\mathrm{spec}(U_{\sigma,\sigma,\sigma}) = \{e^{-\pi i/8}, e^{7\pi i/8}\}$, $\mathrm{spec}(U_{\sigma,1,\sigma}) = \{e^{-\pi i/8}, e^{3\pi i/8}\}$, $\mathrm{spec}(U_{\sigma,\psi,\sigma}) = \{e^{-5\pi i/8}, e^{7\pi i/8}\}$

$p \geq 2$ (even):



$\Rightarrow$ **Poincaré inequality** with $\beta_{p\geq 0} = 1/8$

Outlooks and further references

Summary & outlook

- In 2D, intermediate **exchange** quantum statistics is possible ($e^{i\alpha\pi}$, "anyons").
- **Exchange** $\overset{?}{\leftrightarrow}$ **exclusion** difficult mathematical problem!
- Precise DFT in an **almost-bosonic** limit $\alpha \to 0$, $N \to \infty$, $\alpha N \to \beta \in \mathbb{R}$; also including self interactions $\gamma \in \mathbb{R}$.
- Linearly increasing **stability** for $|\beta| \geq 2$: $\boxed{\gamma \geq -2\pi|\beta|}$
- For $\beta \in 2\mathbb{Z}$ saturated exactly by a manifold of soliton states of dimension $2|\beta|$ ("nonlinear Landau levels").

Ongoing work:

- Derivation from MBQM at $\gamma \neq 0$.
- External fields, potentials & more numerics.
- Fractional $\beta/2$, local sections, fermionic limit...

Almost-fermionic anyons

Similar approach close to fermions: $\alpha = 1 - \beta/N \to 1$, $\hbar \sim N^{-1/2}$.
Also "virtually" **extended anyons**, $0 < R \sim N^{-\eta} \to 0$.
$\Rightarrow$ Actual Thomas–Fermi functional for fermions remains relevant:

$$\mathcal{E}^{\mathrm{TF}}[\varrho] := \int_{\mathbb{R}^2} \left(2\pi \varrho^2 + V \varrho \right)$$

More precisely, a semi-classical **Vlasov functional:**

$$\mathcal{E}^{\mathrm{Vla}}[\mu] := (2\pi)^{-2} \int_{\mathbb{R}^4} \left| \mathbf{p} + \beta \mathbf{A}[\varrho] \right|^2 \mu(\mathbf{x}, \mathbf{p}) d\mathbf{x} d\mathbf{p} + \int_{\mathbb{R}^2} V \varrho \, d\mathbf{x}$$

for $0 \leq \mu(\mathbf{x}, \mathbf{p}) \leq 1$ a measure on phase space $\mathbb{R}^4$, $\int_{\mathbb{R}^4} \mu = (2\pi)^2$.
Minimizer:

$$\mu(\mathbf{x}, \mathbf{p}) = \mathbb{1} \left(|\mathbf{p} + \beta \mathbf{A}[\varrho](\mathbf{x})|^2 \leq 4\pi \varrho(\mathbf{x}) \right)$$

with **spatial density** *independent* of β:

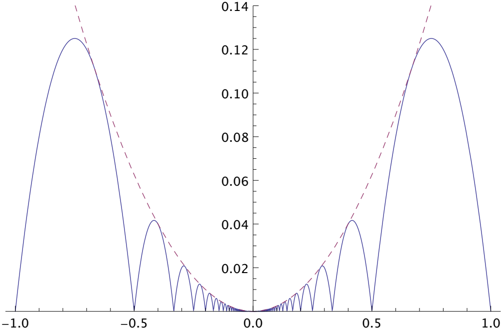
$$\varrho(\mathbf{x}) = (2\pi)^{-2} \int_{\mathbb{R}^2} \mu(\mathbf{x}, \mathbf{p}) d\mathbf{p} = (4\pi)^{-1} (\lambda^{\mathrm{TF}} - V(\mathbf{x}))_+$$

but **momentum density** dep. on β. Chitra, Sen '92; Girardot, Rougerie '21

Intermediate anyons? Try magnetic TF theory

Fermi sea of the Landau Hamiltonian in LDA with a self-generated field $B(\mathbf{x}) = 2\pi\beta\varrho(\mathbf{x})$, $\varrho = \sum_n \varrho_n$, $0 \le \varrho_n \le |B|/(2\pi)$:

$$\mathcal{E}^{\mathrm{mTF}}[\varrho] := \int_{\mathbb{R}^2} \sum_{n=0}^{\infty} \Big(|B|(2n+1)\varrho_n + V\varrho_n \Big) \ge \int_{\mathbb{R}^2} \Big(2\pi(1+M(\beta))\varrho^2 + V\varrho \Big)$$



$$M(\beta) := \beta^2(1 - \{\beta^{-1}\})\{\beta^{-1}\} \ \in [0, \beta^2/4)$$

Girardot, Levitt, Rougerie '21; DL '23; Levitt, DL, Rougerie '25; cf. Chen et al.'89, Hu et al.'21

A possible conjecture for the exact ground-state energy

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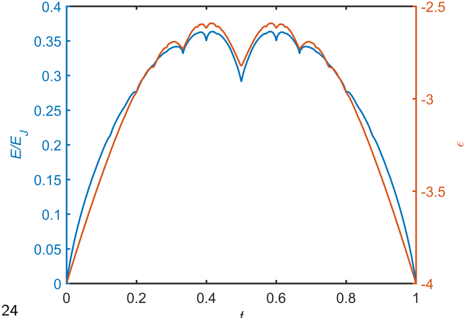
In Figure 6 the lowest energy states of the frustrated XY model are compared to the lowest energy branch of the Hofstadter butterfly which is proportional to the T_c for superconducting networks as predicted by the linearized GL network equations [18–21]. The general shape of the curve is very similar and both show fractal behavior with apparent singularities at rational values of f. However, these singularities seem of different type. In the frustrated XY model these singularities appear logarithmic, while in the Hofstadter butterfly the singularity is approached linearly from both sides. The similarity of these curves suggests there might be a connection between the ground state energy of JJAs and the T_c for superconducting networks.



Lankhorst et al. '18; DL '24

Why 2(+1)D?

"It is the purpose of our research program to study in three-dimensional space-time the classical and quantum motions of matter that interacts gravitationally. Since there are no propagating gravitational degrees of freedom, the problem is tractable, and we can learn much about the puzzles that are encountered when a geometrical theory is confronted by quantum mechanics. In four dimensions these puzzles exist as well, and it is my opinion that understanding them is important for understanding quantum gravity; a task quite independent of and perhaps more fundamental than the task of overcoming the unrenormalizable infinities that pollute four-dimensional gravity, but are absent in three dimensions since non-renormalizable graviton exchange does not occur."

Jackiw, *Topics in planar physics*, 1990

Quantum gravity: anyons on the event horizon...

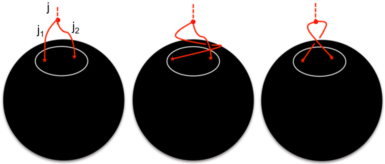


FIG. 3 (color online). Two incident bulk edges piercing the horizon: unbraided vs upon the application of $\hat{M}$ and $\hat{B}$, respectively.

knotting of the spin network at least in the vicinity of the horizon as in Fig. 3, which cannot be unraveled through a (small) bulk diffeomorphism. In the following we want to investigate whether such a different knotting of the spin network in the neighborhood of the horizon has any observable consequences. The area operator would not be of great help here, since $\hat{A}$ is a function of the $su(2)$-Casimir operator and thus commutes with all the generators of this Lie algebra. For a representation $\hat{\rho}$ of a generic element of the braid group one has

$$\langle \hat{\rho}\psi | \hat{A} | \hat{\rho}\psi \rangle = \langle \psi | \hat{\rho}^{-1} \hat{A}\, \hat{\rho}\, | \psi \rangle = \langle \psi | \hat{A} | \psi \rangle. \tag{61}$$

unless $n = 2$ and $k \to \infty$. For $n = 1$ and $k \to \infty$ (65) reduces to expression (43). For example, when $n = 2$ the commutator yields

$$\begin{aligned} [\hat{F}^i, \hat{M}] = {} & i \frac{4\pi}{k+2} \frac{4\pi}{k} \\ & \times i\epsilon^i_{jk} (\delta^2(x, x_1) \hat{J}^k_{\rho_1} \otimes \hat{J}^j_{\rho_2} + \hat{J}^j_{\rho_1} \otimes \hat{J}^k_{\rho_2} \delta^2(x, x_2)) \\ & + \mathcal{O}(k^{-3}). \end{aligned} \tag{66}$$

A local stationary observer who resides on the node in Fig. 3 at proper distance ℓ to the horizon will be able to discern braided from unbraided states e.g. by measuring differences in the expectation values of the field strength operator. When considering large black holes the effect of the braiding onto the field strength would be negligible but it would become relevant for smaller (and smaller getting) black holes.

The physical picture behind the statistical phase is very similar to what happens in electromagnetism when dealing with the Aharonov-Bohm effect. To see this we use the ideas presented in [49] and consider a locally flat connection on $S^2 - \{p\}$

$$A_{i\varphi}(x) = \frac{\phi_i}{\ } \alpha_{\varphi}(x) \tag{67}$$

Some math-phys references on the 2D anyon gas

D. L., *Properties of 2D anyon gas*, invited contribution to the Encyclopedia of Condensed Matter Physics 2e, 2024, `arXiv:2303.09544`

See also Leinaas & Myrheim, Fröhlich, and others in the same volume.

D. L., V. Qvarfordt, *Exchange and exclusion in the non-abelian anyon gas*, 2020/2026, `arXiv:2009.12709`

Lecture notes for a master-level course: `arXiv:1805.03063`,
`http://www.mathematik.uni-muenchen.de/~lundholm/methmmp.pdf`

T. Girardot, *Mean-field approximation for the anyon gas*, Ph.D. thesis, Université Grenoble Alpes & CNRS, 2021.

D. L., *2D magnetic stability*, Int. J. Geom. Meth. Mod. Phys. 2026, `arXiv:2410.24156`

A. Ataei, D. L., T. Girardot, *Microscopic derivation of the stationary Chern-Simons-Schrdinger equation for almost-bosonic anyons*, 2025, `arXiv:2504.17488`

A. Ataei, A. Ellingsen, F. Getzner, T. Girardot, D. L., and D.-T. Nguyen, *Nonlinear Landau levels in the almost-bosonic anyon gas*, 2025, `arXiv:2510.14679`