Geometric Topology and Textures in Soft Matter

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Lecture 3: Hopfions and Chiral Topology

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photo: Michi Nakata

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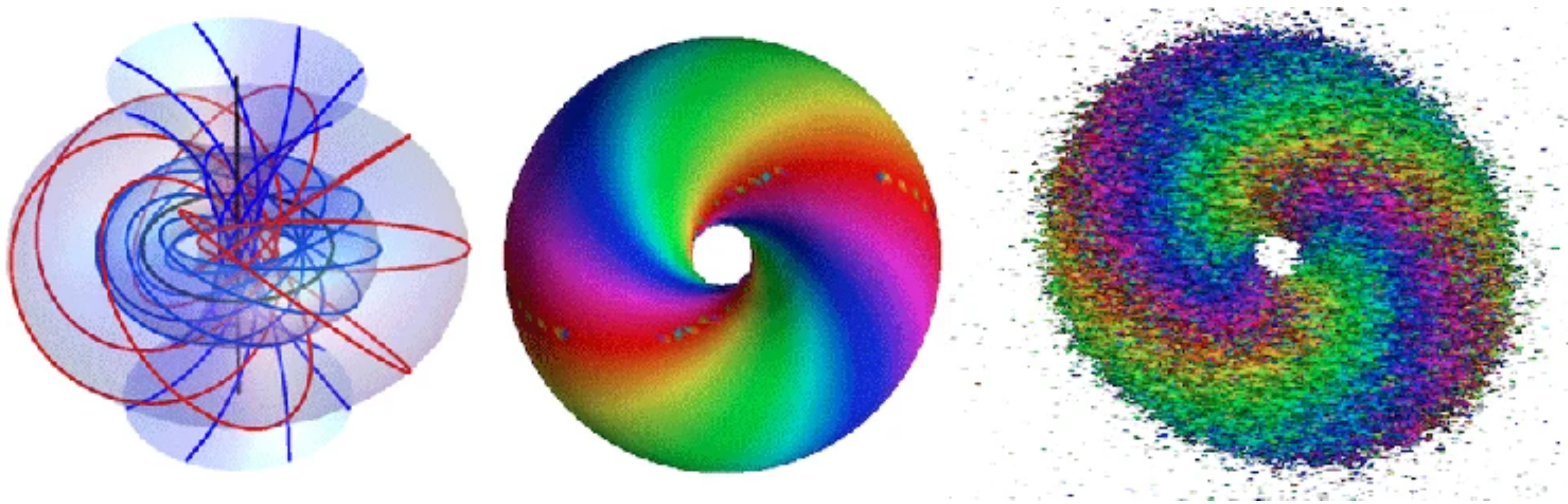
I. The Hopf Fibration

A **hopfion** is a *topologically non-trivial*, and *non-singular*, *three-dimensional texture*. They can be thought of as realisations of the famous **Hopf map** between S^3 and S^2, which is the generator of $\pi_3(S^2) \cong \mathbb{Z}$.



B. G. Chen *et al.*
Phys. Rev. Lett. **110**, 237801 (2013).

I. The Hopf Fibration

The Hopf map can be described briefly using complex coordinates. Points of S^3 are given as $(z_1, z_2) \in \mathbb{C}^2 \cong \mathbb{R}^4$ with $|z_1|^2 + |z_2|^2 = 1$. Points of S^2 are given as a complex coordinate by stereographic projection. The **Hopf map** is

$$(z_1, z_2) \mapsto \zeta = \frac{z_1}{z_2}\,.$$

Concretely, this gives a director field with Cartesian components

$$n_x + i n_y = 2\, z_1 \bar{z}_2, \qquad n_z = |z_1|^2 - |z_2|^2\,.$$

Although direct, this is moderately abstract. It also gives the texture on S^3 rather than $\mathbb{R}^3$. This can be overcome by again using stereographic projection, but then the texture fills all of $\mathbb{R}^3$ and is not localised as in the experiments.

We give below a direct construction of hopfions as localised objects in $\mathbb{R}^3$, but first we describe the structure of the **Hopf fibration**. For this, we use the Pontrjagin-Thom construction.

I. The Hopf Fibration

$$n_x + i n_y = 2\, z_1 \bar{z}_2, \qquad n_z = |z_1|^2 - |z_2|^2 .$$

We describe the structure of the **Hopf fibration** using the Pontrjagin-Thom construction.

The magnitudes of both z_1 and z_2 can be written in terms of $n_z = \cos\theta$

$$|z_1|^2 = \frac{1+n_z}{2}, \qquad |z_2|^2 = \frac{1-n_z}{2} .$$

So $z_1 = \cos\frac{\theta}{2}\,\mathrm{e}^{iu}$, $z_2 = \sin\frac{\theta}{2}\,\mathrm{e}^{iv}$ and the inverse image of all orientations with a fixed θ is topologically a torus, $T^2 = S^1 \times S^1$.

Then $n_x + i n_y = \sin\theta\,\mathrm{e}^{i(u-v)}$ and for any azimuthal angle ϕ we have $u = \eta + \phi/2$, $v = \eta - \phi/2$; the inverse image of any orientation (θ, ϕ) is a **circle**, parameterised by η.

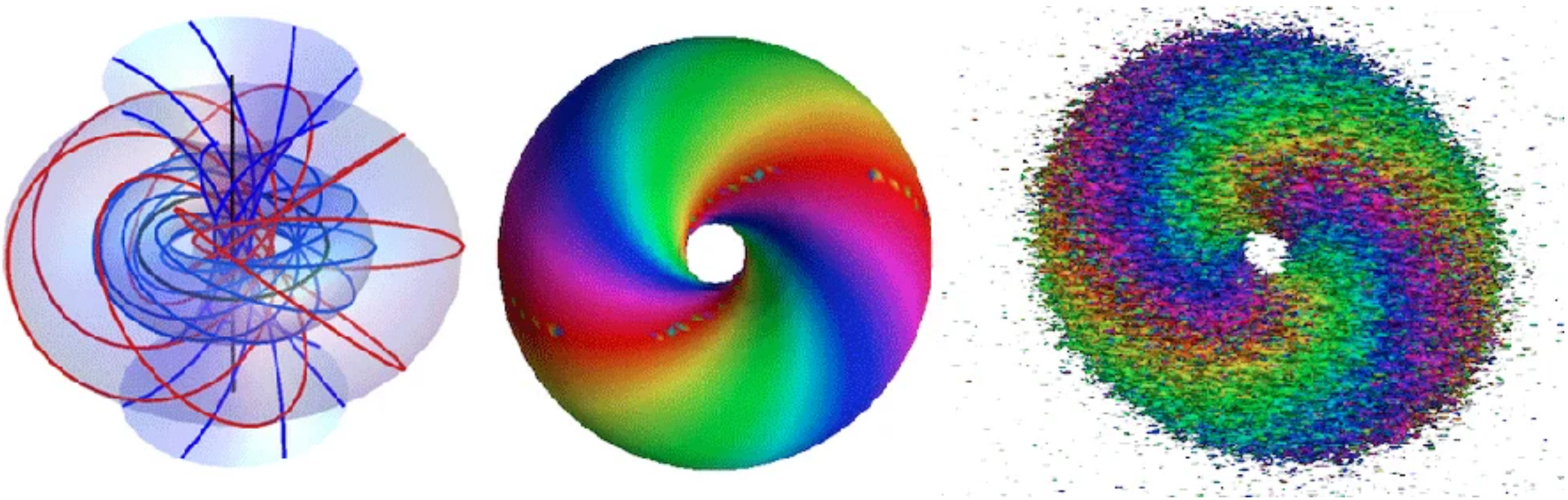
For example, the inverse image of $\mathbf{n} = \mathbf{e}_x$ is the circle $(\mathrm{e}^{i\eta}/\sqrt{2}, \mathrm{e}^{i\eta}/\sqrt{2})$.

The structure of the Hopf map is that of a **fibration** $S^1 \to S^3 \to S^2$. The key feature is that the fibres (inverse image circles) are **all linked** with each other. Their **linking number**, called the **Hopf invariant**, is a topological label for hopfion states.

Exercise: Plot this!

I. The Hopf Fibration

The structure of the Hopf map is that of a **fibration** $S^1 \to S^3 \to S^2$. The key feature is that the fibres (inverse image circles) are **all linked** with each other. Their **linking number**, called the **Hopf invariant**, is a topological label for hopfion states.



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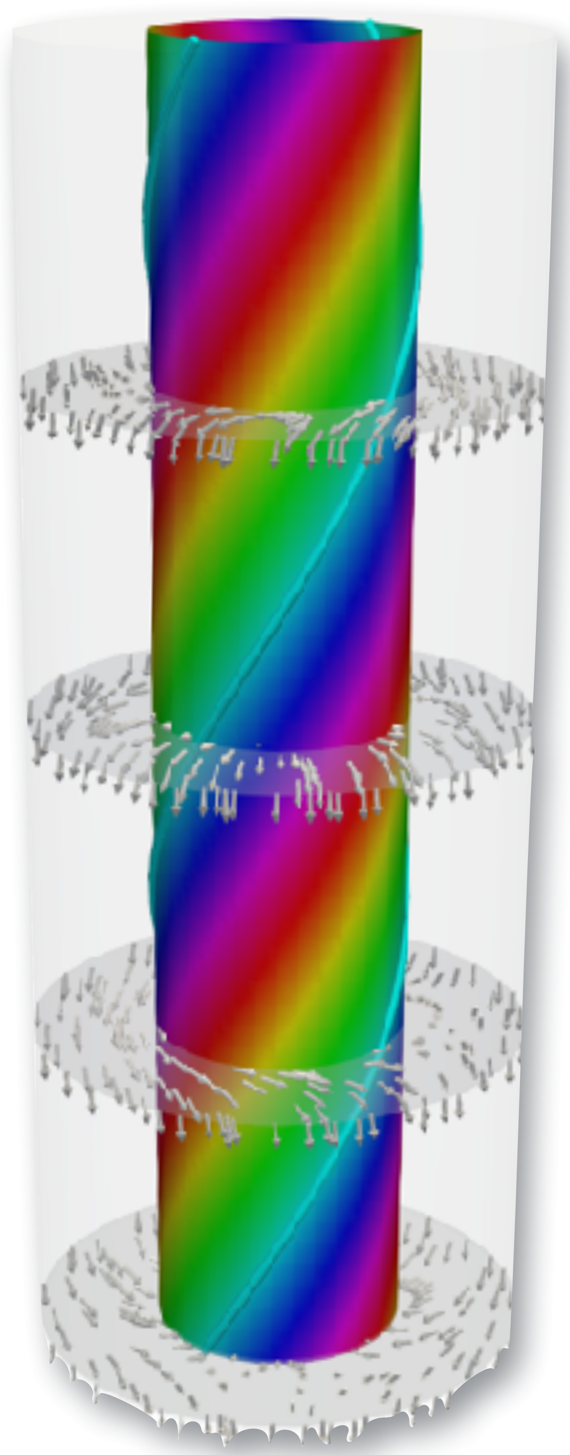
II. Generation of Hopfions in $\mathbb{R}^3$



Recall the skyrmion texture on a disc of radius R

$$\mathbf{n} = \cos\frac{\pi r}{R}\,\mathbf{e}_z + \sin\frac{\pi r}{R}\left(\cos(\phi-\alpha)\,\mathbf{e}_x + \sin(\phi-\alpha)\,\mathbf{e}_y\right),$$

where (r,ϕ) are polar coordinates for the disc and α is a free parameter.

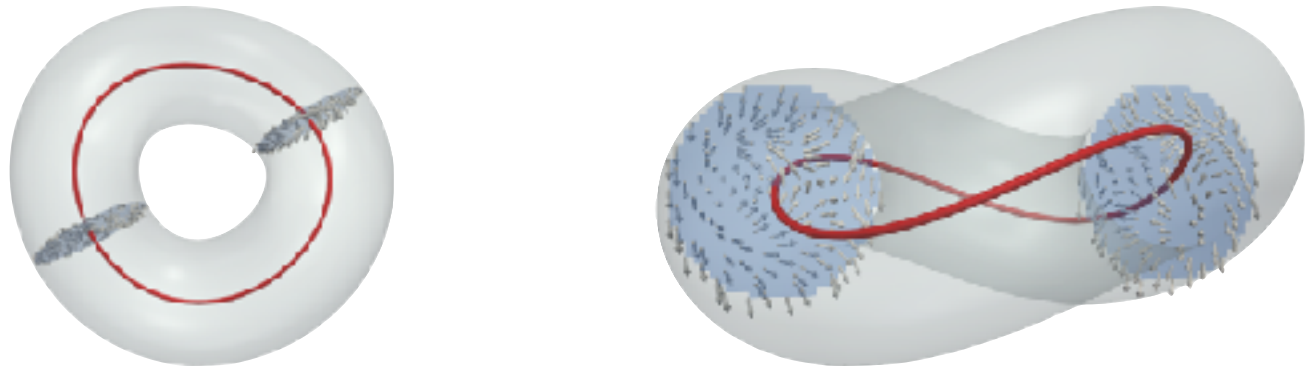
We generalise this to a texture on a solid cylinder $D^2 \times [0,L]$ of length L and let $\alpha(s)$ be a function of the coordinate s on the last factor.

If the skyrmion texture on the two ends is the same then $\alpha(L) = \alpha(0) + 2\pi h$ for an integer h. This integer is the number of full **twists** in the skyrmion cylinder and will ultimately be the **Hopf invariant** of the final hopfion texture.

The Pontrjagin-Thom construction displays this (the inverse image of all 'horizontal' orientations) as a 'barber pole'.

II. Generation of Hopfions in $\mathbb{R}^3$

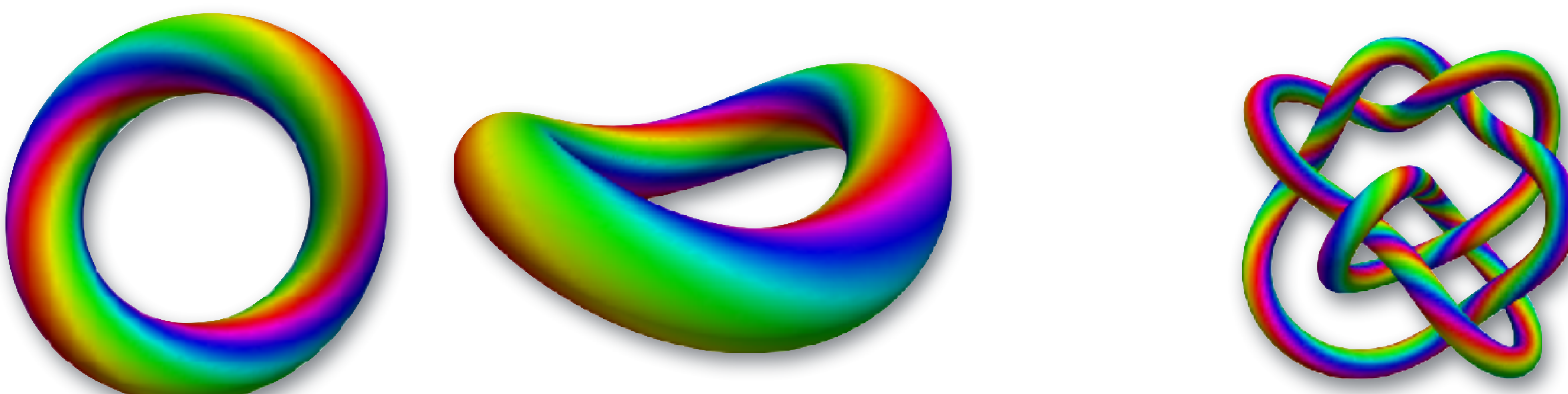
This construction can be applied to the **tubular neighbourhood** of a closed curve K, identified as the axis of the cylinder.



s is arclength along K; r is radial distance away from it; for the azimuthal angle we can take the **solid angle** of the curve (divided by two); and α is any degree h map $K \to S^1$.

This produces a texture where the inverse images (of regular values) are all topologically circles (equivalent to K) that are linked with one another with **linking number** h.

For the basic experimental hopfions K is a **circular unknot** but the same construction works when K is **any knot**, or **link**.

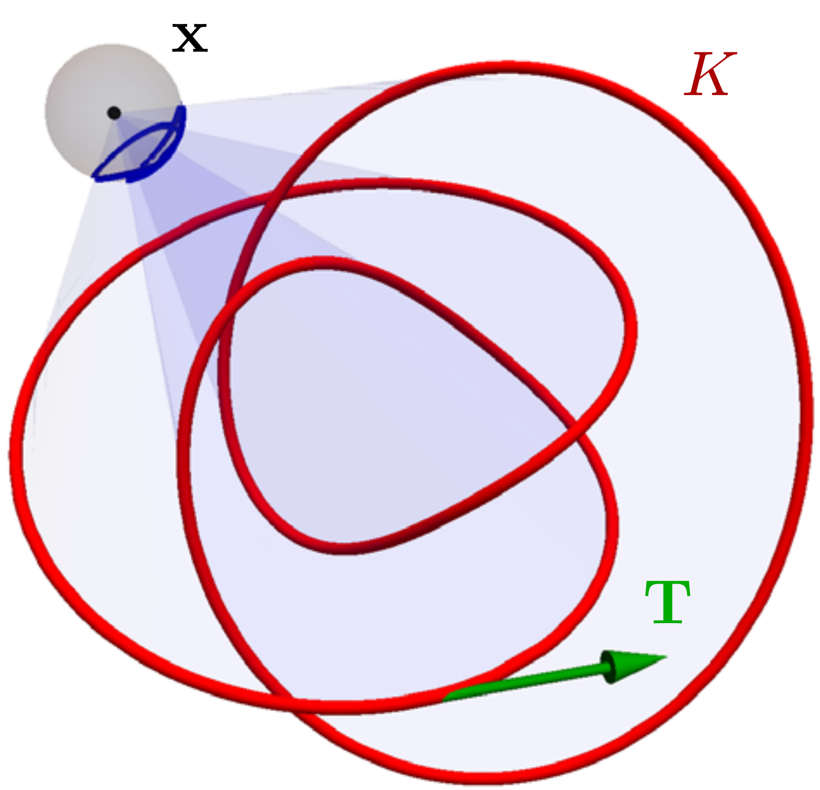


9^2_{50}

III. On Solid Angle

The **solid angle** gives an angle winding around any closed curve in $\mathbb{R}^3$. A wonderful account of its properties was given by Maxwell in his *Treatise on Electricity and Magnetism* — it is proportional to the magnetostatic potential of a current-carrying wire.

He also gave purely geometric descriptions. Let K be a closed curve and $\mathbf{x}$ a point in its complement. The solid angle $\omega(\mathbf{x}; K)$ is the area bound by the projection of K onto the unit sphere centred at $\mathbf{x}$. It is defined modulo 4π.



current carrying wire:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \qquad \nabla \cdot \mathbf{B} = 0.$$

outside the wire:

$$\mathbf{B} = \nabla \omega, \qquad \nabla^2 \omega = 0, \qquad (\omega \text{ is harmonic})$$

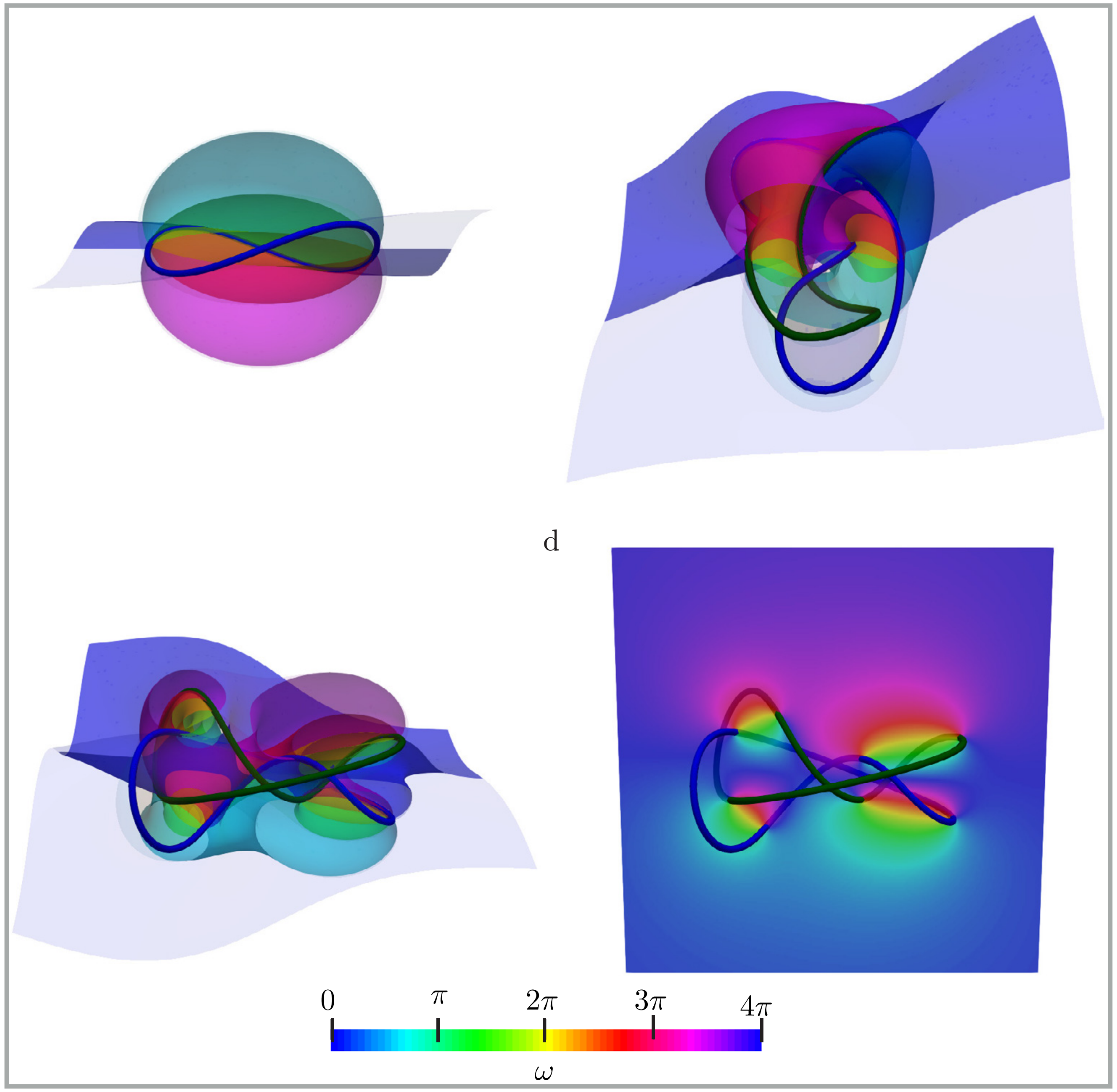
$$\int_C d\omega = 4\pi \; (= \mu_0 I)$$

meridian of K

James Clerk Maxwell

III. On Solid Angle

Maxwell gave three separate (and practical) geometric methods for computing the solid angle.



Binysh, GPA, J. Phys. A: Math. Theor. **51**, 385202 (2018)

1. $$\omega(\mathbf{x}) = \int_K \frac{\mathbf{v}_\infty \times \mathbf{v} \cdot d\mathbf{v}}{1 + \mathbf{v}_\infty \cdot \mathbf{v}}$$

   v — direction of *K* from **x**

2. $$\omega(\mathbf{x}) = 2\pi(1 + \mathrm{Wr}) - \int_K \frac{\mathbf{v} \cdot \mathbf{T} \times d\mathbf{T}}{1 + \mathbf{v} \cdot \mathbf{T}}$$

   T — tangent to *K*

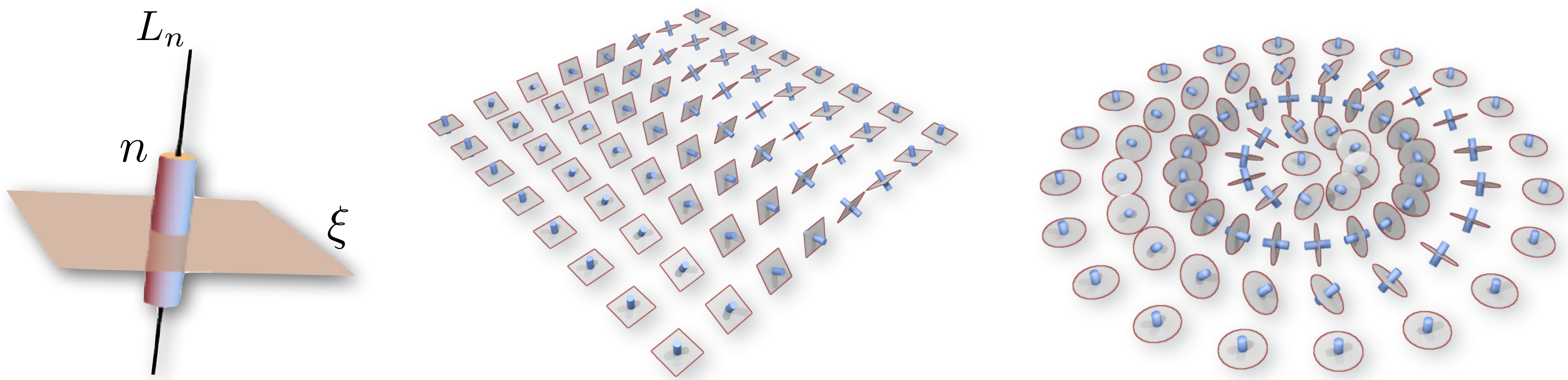
3. $$\omega_{K_1}(\mathbf{x}) - \omega_{K_0}(\mathbf{x}) = \int \frac{\mathbf{v}_0 \times \mathbf{v}_1 \cdot d(\mathbf{v}_0 + \mathbf{v}_1)}{1 + \mathbf{v}_0 \cdot \mathbf{v}_1}$$

   homotopy formula

James Clerk Maxwell

IV. Geometry of Director Fields

In three dimensions, the director field splits directions in space into those **parallel** to it and those **perpendicular**. We call the perpendicular (or orthogonal) plane ξ.



At every point there is a **local symmetry group** consisting of rotations that preserve the director field; it is isomorphic to $SO(2)$. The action of the local symmetry group gives a decomposition of the director gradients into pieces that transform separately [Machon, GPA (2016)]

$$\nabla \mathbf{n} = \mathbf{n} \otimes \nabla_{\mathbf{n}} \mathbf{n} + \frac{\nabla \cdot \mathbf{n}}{2} \mathbf{I}_{\xi} - \frac{\mathbf{n} \cdot \nabla \times \mathbf{n}}{2} \mathbf{J}_{\xi} + \mathbf{\Delta} \,.$$

bend ***splay*** ***twist*** ***'deviatoric'***

IV. Geometry of Director Fields

The structure of the gradient matrix at any point can be given in an adapted frame where $\mathbf{n} = \mathbf{e}_z$ and ξ is the xy-plane

$$\nabla \mathbf{n} = \left[\begin{array}{cc|c} \partial_x n_x & \partial_x n_y & 0 \\ \partial_y n_x & \partial_y n_y & 0 \\ \hline \partial_z n_x & \partial_z n_y & 0 \end{array}\right].$$

The bottom row contains the **parallel gradients** $\nabla_{\|}\mathbf{n} = \mathbf{n} \otimes \nabla_{\mathbf{n}}\mathbf{n}$; the vector $\nabla_{\mathbf{n}}\mathbf{n}$ is called **bend**.

The upper-left 2x2 block is the **orthogonal gradients** $\nabla_{\perp}\mathbf{n}$; they further decompose as

$$\nabla_{\perp}\mathbf{n} = \frac{\nabla \cdot \mathbf{n}}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{\mathbf{n} \cdot \nabla \times \mathbf{n}}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \frac{\partial_x n_x - \partial_y n_y}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{\partial_x n_y + \partial_y n_x}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

These are **splay** ($\nabla \cdot \mathbf{n}$), **twist** ($\mathbf{n} \cdot \nabla \times \mathbf{n}$) and the **deviatoric gradients** ($\boldsymbol{\Delta}$).

$$\nabla \mathbf{n} = \mathbf{n} \otimes \nabla_{\mathbf{n}}\mathbf{n} + \frac{\nabla \cdot \mathbf{n}}{2}\mathbf{I}_{\xi} - \frac{\mathbf{n} \cdot \nabla \times \mathbf{n}}{2}\mathbf{J}_{\xi} + \boldsymbol{\Delta}$$

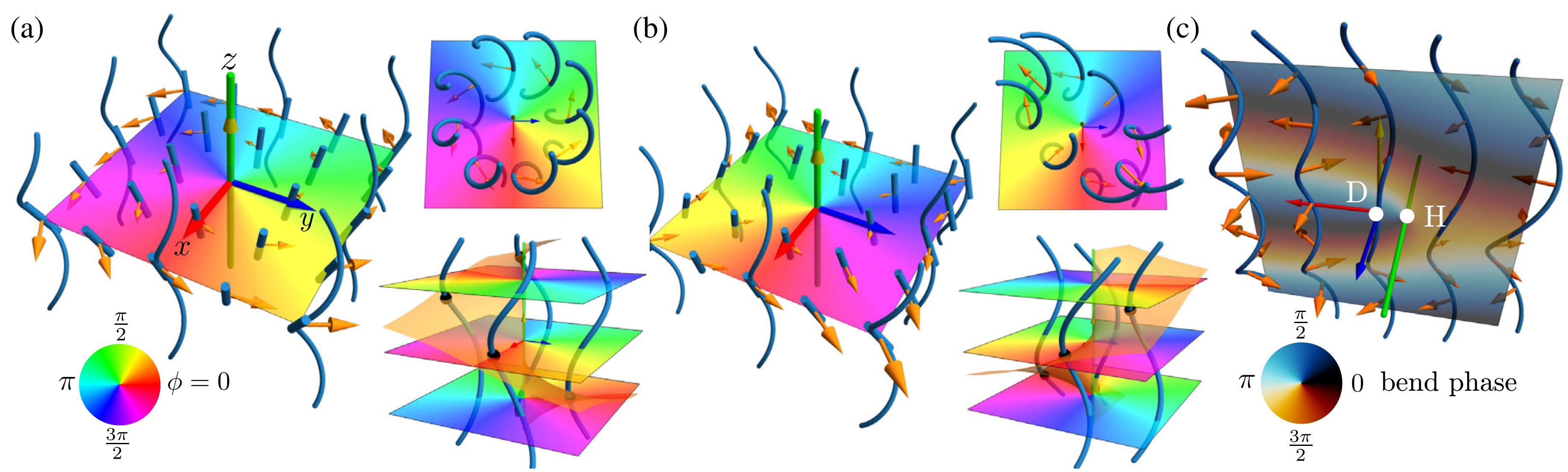
bend *splay* *twist* *'deviatoric'*

IV. Geometry of Director Fields

Each component — **bend** ($\nabla_{\mathbf{n}}\mathbf{n}$), **splay** ($\nabla \cdot \mathbf{n}$), **twist** ($\mathbf{n} \cdot \nabla \times \mathbf{n}$), **deviatoric gradients** ($\boldsymbol{\Delta}$) — carries its own geometry.

Loosely, bend is the geometry of curves; the integral curves of the director. And when twist vanishes, splay and the deviatoric gradients are the geometry of surfaces; the integral surfaces of the planes perpendicular to the director.

Both bend ($\nabla_{\mathbf{n}}\mathbf{n}$) and the deviatoric gradients ($\boldsymbol{\Delta}$) convey an orientation that lies entirely in the orthogonal plane (ξ). It is like a ***'texture within a texture'*** that can be analysed with the same methods of differential geometry as we have described here.



Twist is about **handedness** and is an instance of **contact geometry**. We will describe some of its unique topology. This is measured by the sign of twist $\mathbf{n} \cdot \nabla \times \mathbf{n} = q \neq 0$; it is **right-handed** when q is **negative**.

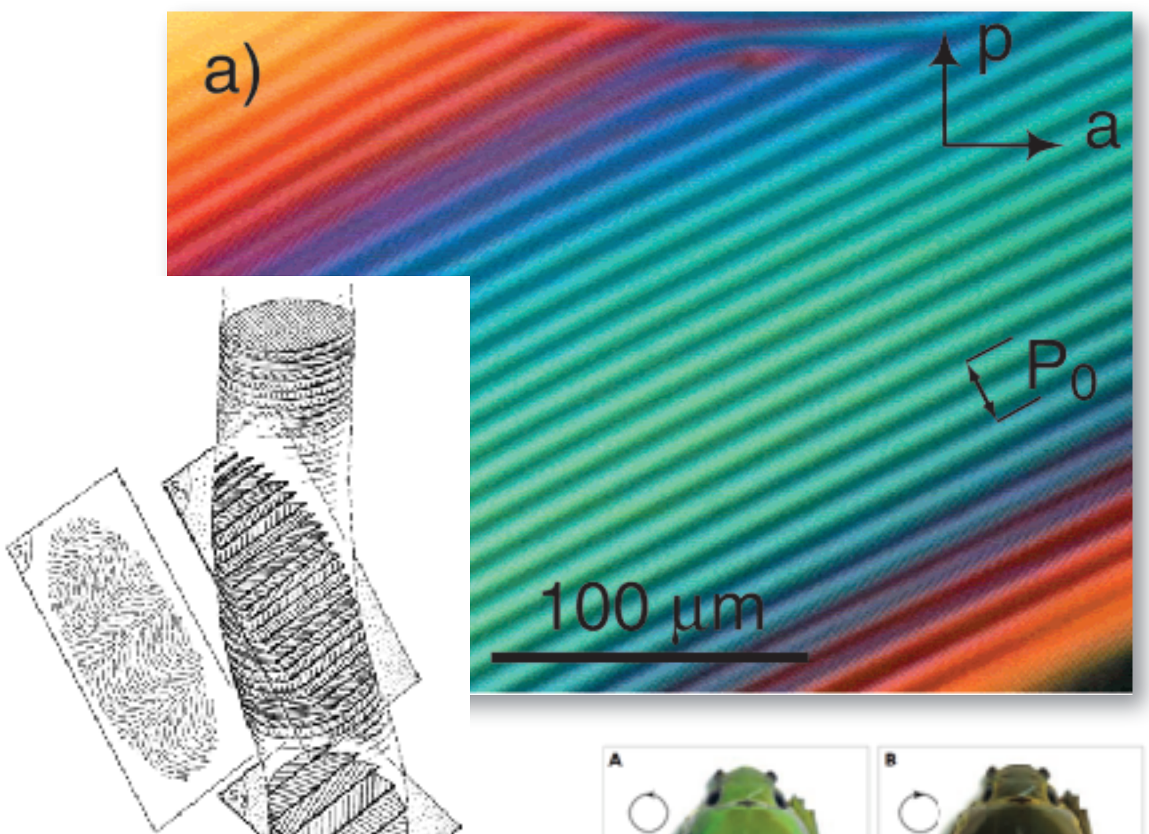
A basic question is whether a topological texture admits a consistent handedness.

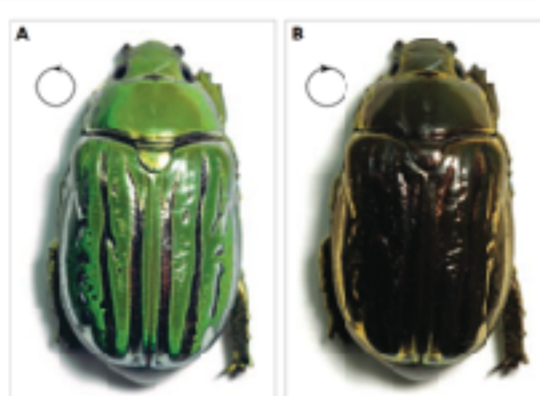
V. Cholesterics: Twist, Pitch Axis and Umbilic Lines

The basic texture with twist is the **cholesteric ground state**, given by the director

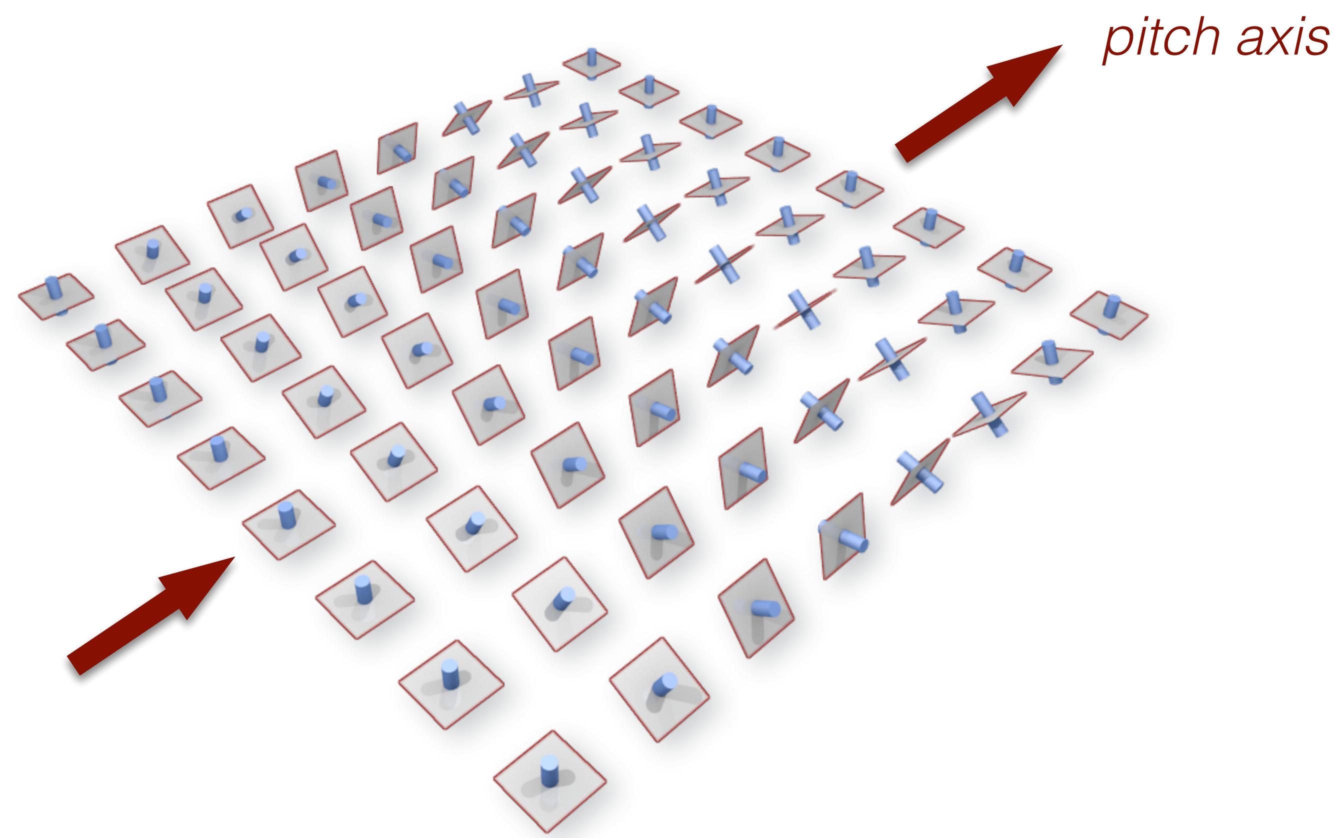
$$\mathbf{n} = \cos q_0 z\, \mathbf{e}_x + \sin q_0 z\, \mathbf{e}_y,$$

or any equivalent to it by a Euclidean motion. It is a helical rotation of the orientation along a direction perpendicular to the director. This is called the **pitch axis** (here the z-axis). The parameter q_0 — called the **chirality** — is the rate of rotation and sets the length scale $p = 2\pi/q_0$ — called the **pitch** — of a full 2π rotation.





C. gloriosa



V. Cholesterics: Twist, Pitch Axis and Umbilic Lines

The cholesteric ground state is the director $\mathbf{n} = \cos q_0 z\, \mathbf{e}_x + \sin q_0 z\, \mathbf{e}_y$.

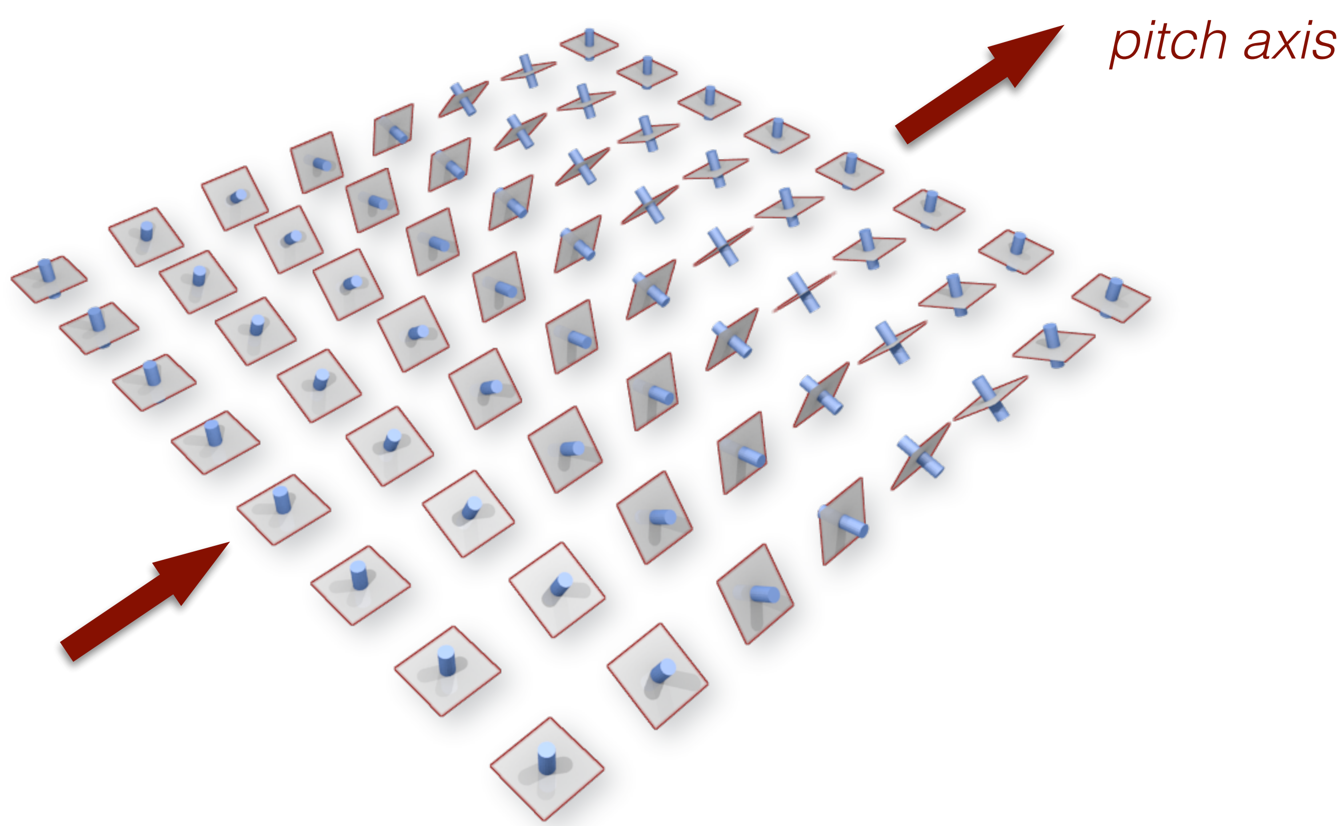
$$\nabla \mathbf{n} = \mathbf{n} \otimes \nabla_{\mathbf{n}} \mathbf{n} + \frac{\nabla \cdot \mathbf{n}}{2} \mathbf{I}_\xi - \frac{\mathbf{n} \cdot \nabla \times \mathbf{n}}{2} \mathbf{J}_\xi + \boldsymbol{\Delta} \, .$$

The gradients of the cholesteric ground state are

$$\nabla \mathbf{n} = q_0\, \mathbf{e}_z \otimes \left(\cos q_0 z\, \mathbf{e}_y - \sin q_0 z\, \mathbf{e}_x \right) \equiv q_0\, \mathbf{e}_z \otimes \mathbf{n}_\perp ,$$

$$= \frac{q_0}{2}\left(\mathbf{e}_z \otimes \mathbf{n}_\perp - \mathbf{n}_\perp \otimes \mathbf{e}_z \right) + \frac{q_0}{2}\left(\mathbf{e}_z \otimes \mathbf{n}_\perp + \mathbf{n}_\perp \otimes \mathbf{e}_z \right) .$$

The state has *no bend*, *no splay*, the *twist* is $\mathbf{n} \cdot \nabla \times \mathbf{n} = -q_0$ and $\boldsymbol{\Delta}$ is *non-zero*.



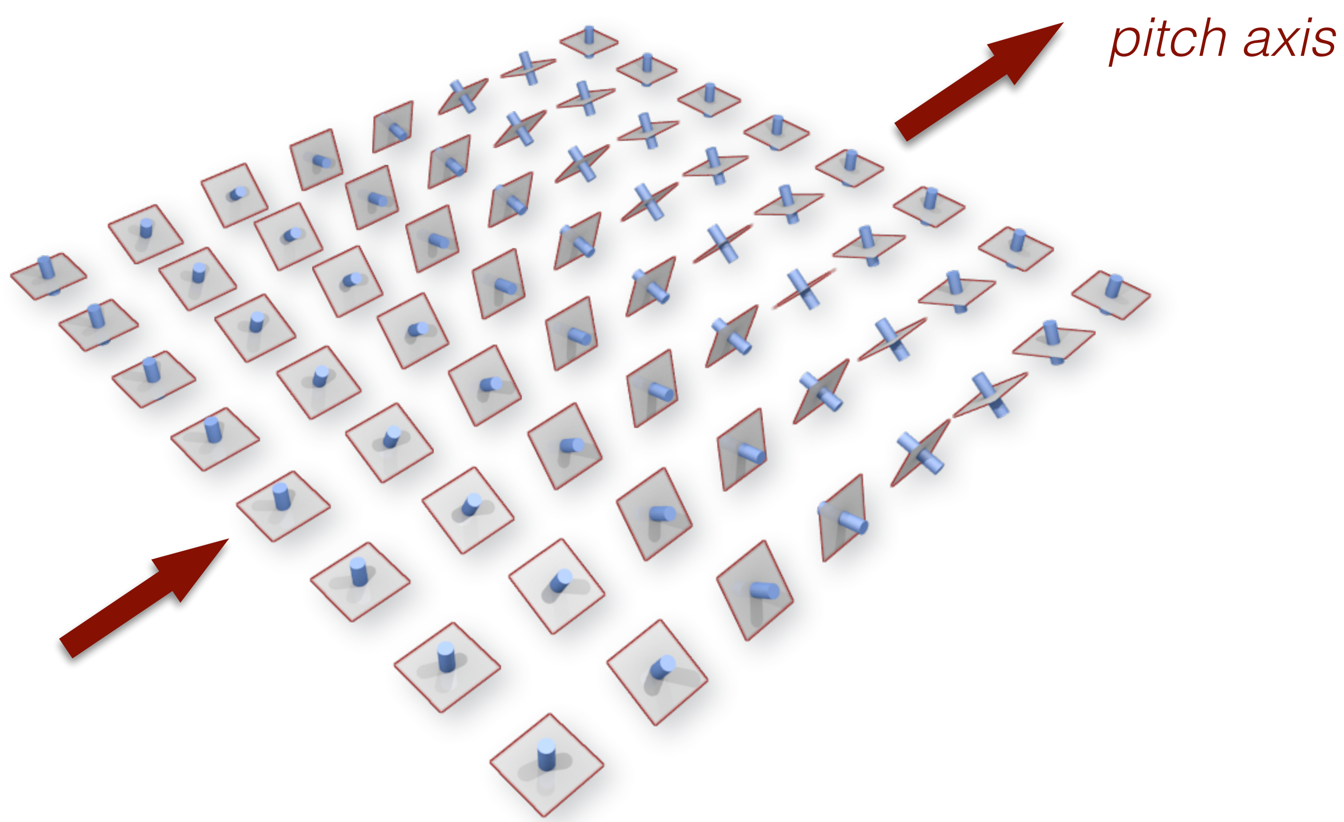
V. Cholesterics: Twist, Pitch Axis and Umbilic Lines

$$\nabla \mathbf{n} = \frac{q_0}{2}\left(\mathbf{e}_z \otimes \mathbf{n}_\perp - \mathbf{n}_\perp \otimes \mathbf{e}_z\right) + \frac{q_0}{2}\left(\mathbf{e}_z \otimes \mathbf{n}_\perp + \mathbf{n}_\perp \otimes \mathbf{e}_z\right).$$

Act with $\mathbf{J}_\xi$ on the second factor to get

$$\frac{q_0}{2}\left(\mathbf{e}_z \otimes \mathbf{e}_z + \mathbf{n}_\perp \otimes \mathbf{n}_\perp\right) + \frac{q_0}{2}\left(\mathbf{e}_z \otimes \mathbf{e}_z - \mathbf{n}_\perp \otimes \mathbf{n}_\perp\right) \equiv \frac{q_0}{2}\mathbf{I}_\xi + \mathbf{J}_\xi \boldsymbol{\Delta}.$$

The **pitch axis** ($\mathbf{e}_z$) can be identified with the *largest eigenvector*† of $\mathbf{J}_\xi \boldsymbol{\Delta}$.

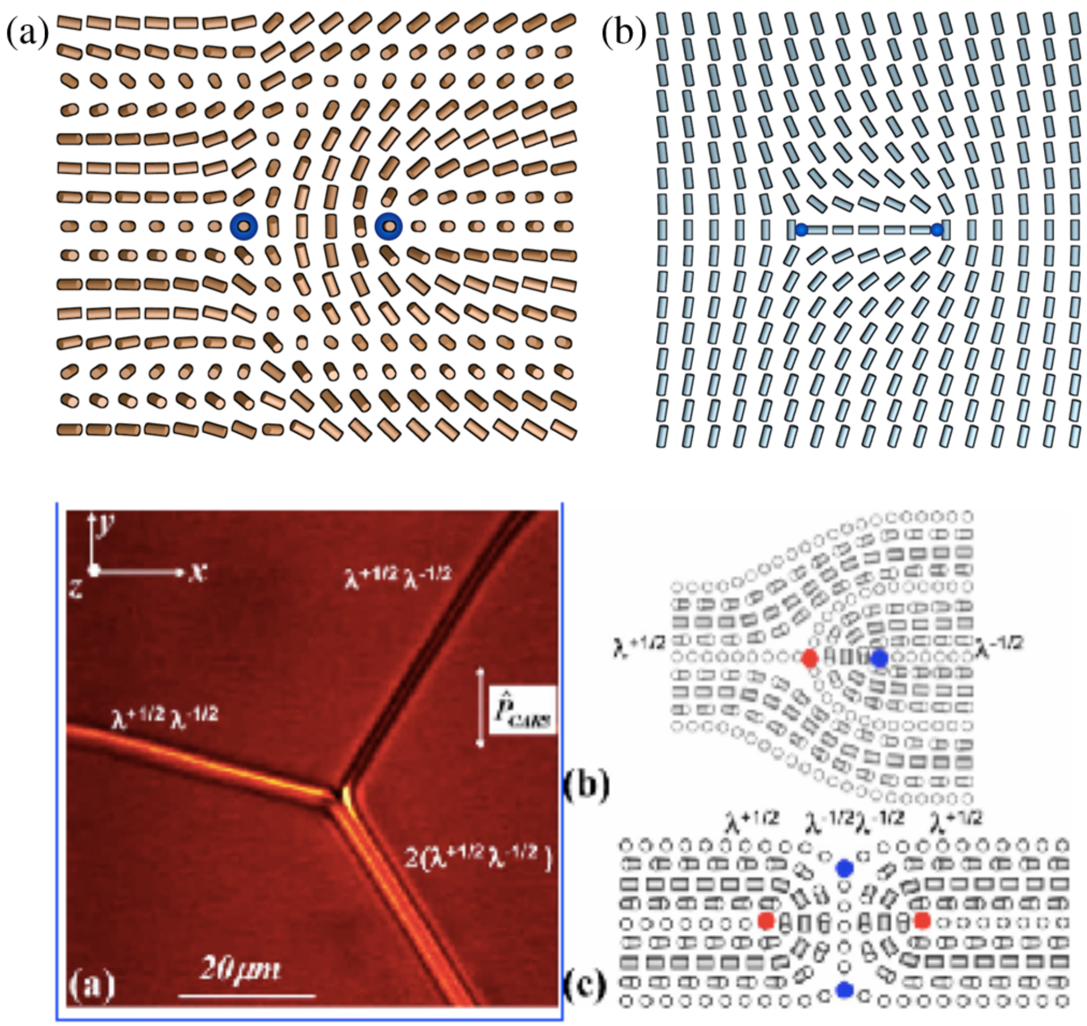


†More correctly, the pitch axis is the eigenvector whose eigenvalue has the same sign as q_0.

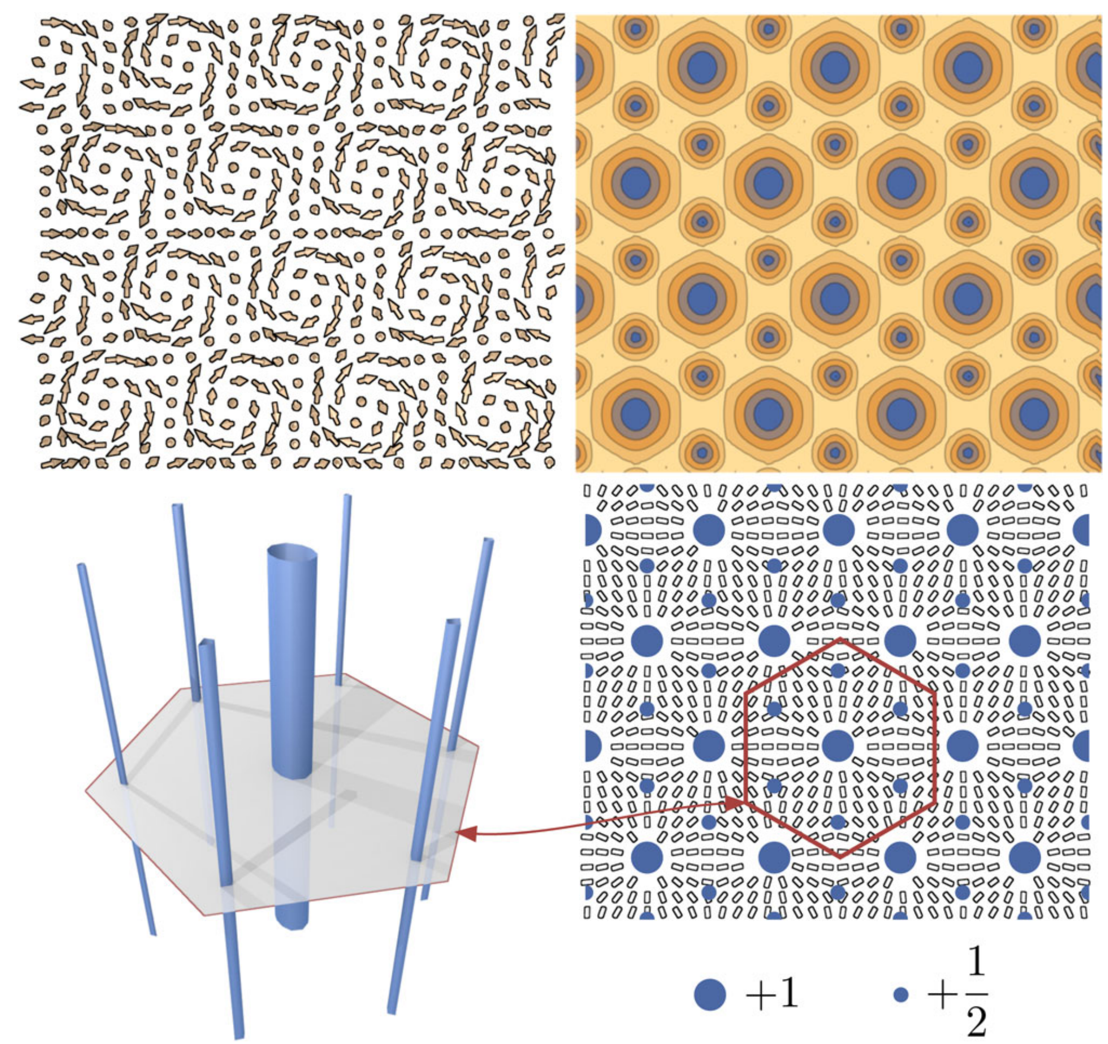
V. Cholesterics: Twist, Pitch Axis and Umbilic Lines

Defects in the pitch axis (*λ lines*) correspond to **zeros** of $\mathbf{J}_\xi\boldsymbol{\Delta}$ — in general we call these **umbilic lines**.

λ lines (cholesterics)



skyrmions



V. Cholesterics: Twist, Pitch Axis and Umbilic Lines

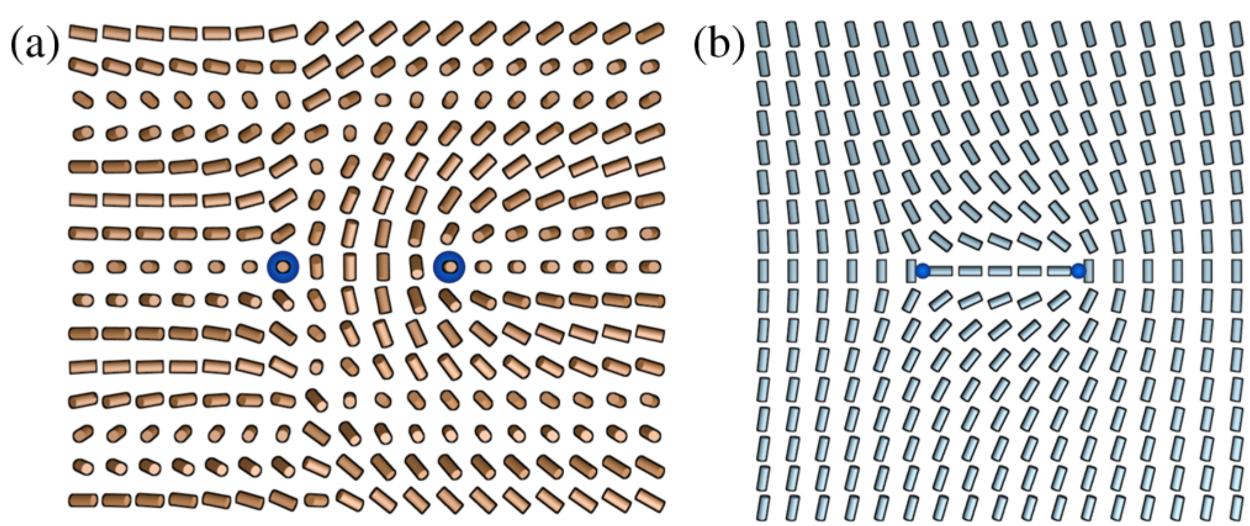
Defects in the pitch axis (λ *lines*) correspond to **zeros** of $\mathbf{J}_\xi\boldsymbol{\Delta}$ — in general we call these **umbilic lines**.

Exercise: A local model for a zero is given by†

$$\mathbf{J}_\xi\boldsymbol{\Delta} \sim \begin{bmatrix} x & y \\ y & -x \end{bmatrix} \sim r\begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{bmatrix}.$$

A way of visualising it is to plot its eigenvectors; show that it looks like a $+\frac{1}{2}$ (nematic) defect. What is a local model for a $-\frac{1}{2}$ defect?

$\boldsymbol{\lambda}$ lines (cholesterics)



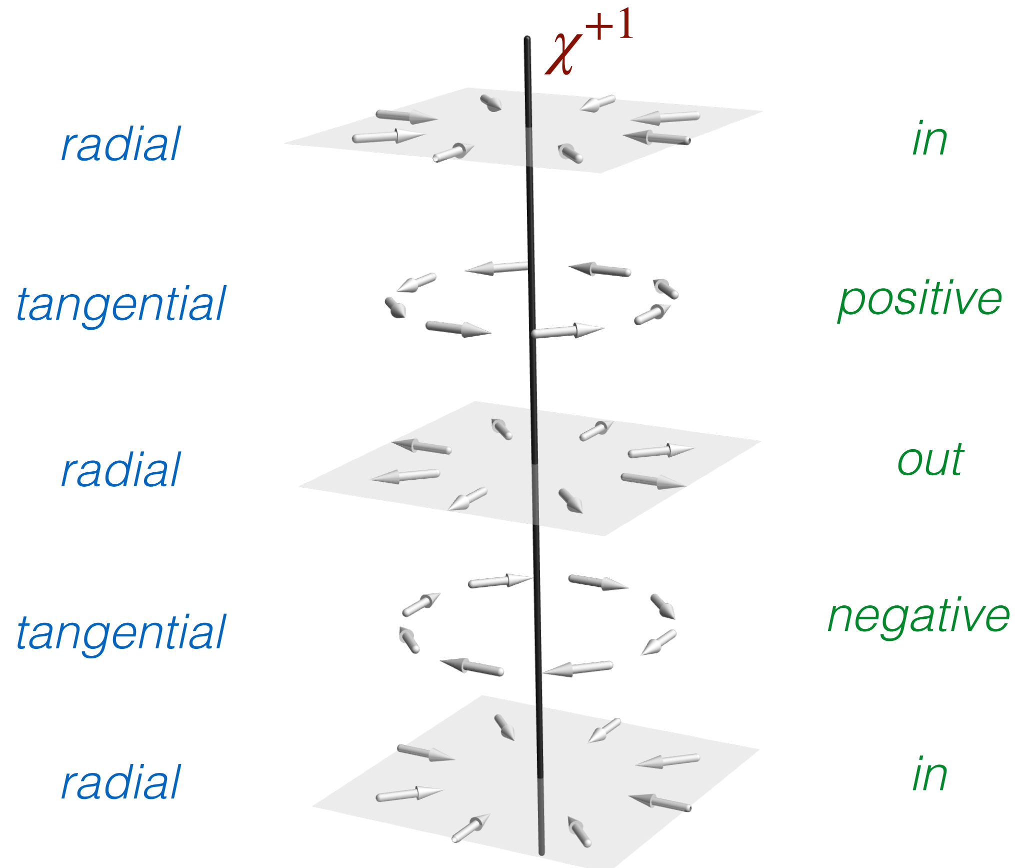
†The basis of this matrix is an orthonormal pair of vectors in the orthogonal plane ξ in some neighbourhood of the origin (location of the defect).

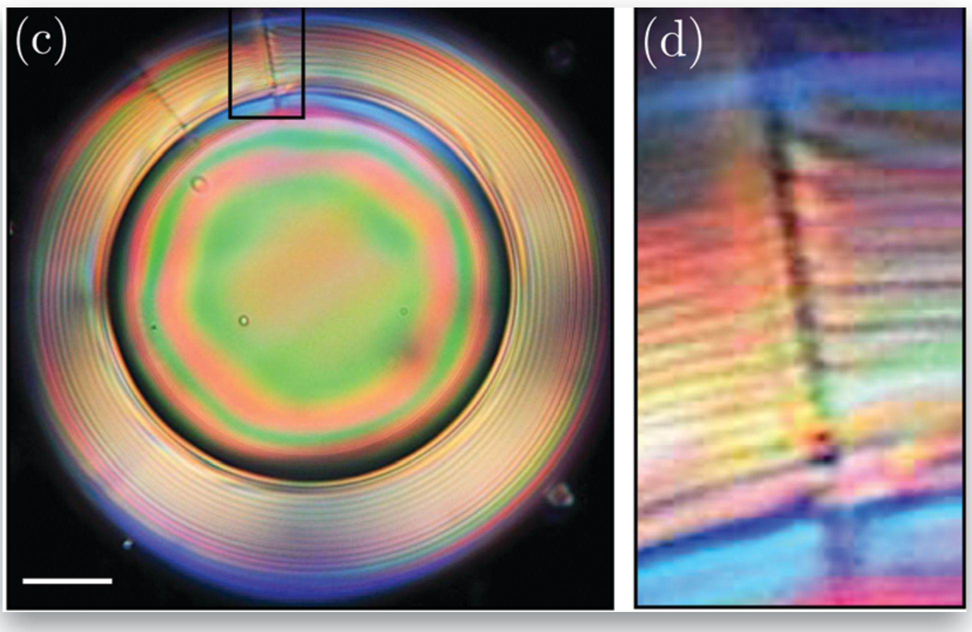
V. Chiral Escape, Frustration and Defects

The director field (with ϕ the azimuthal angle about the z-axis)

$$\mathbf{n} = \cos(q_0 z + \phi)\,\mathbf{e}_x + \sin(q_0 z + \phi)\,\mathbf{e}_y,$$

describes a **chiral** singular line — a χ **line** — with winding number $+1$.





We attempt to remove it, in the style of Meyer, by *escape into the third dimension*. It is an amazing fact of **chiral topology** that this ***cannot be done*** while maintaining the handedness [Pollard, GPA (2024)].

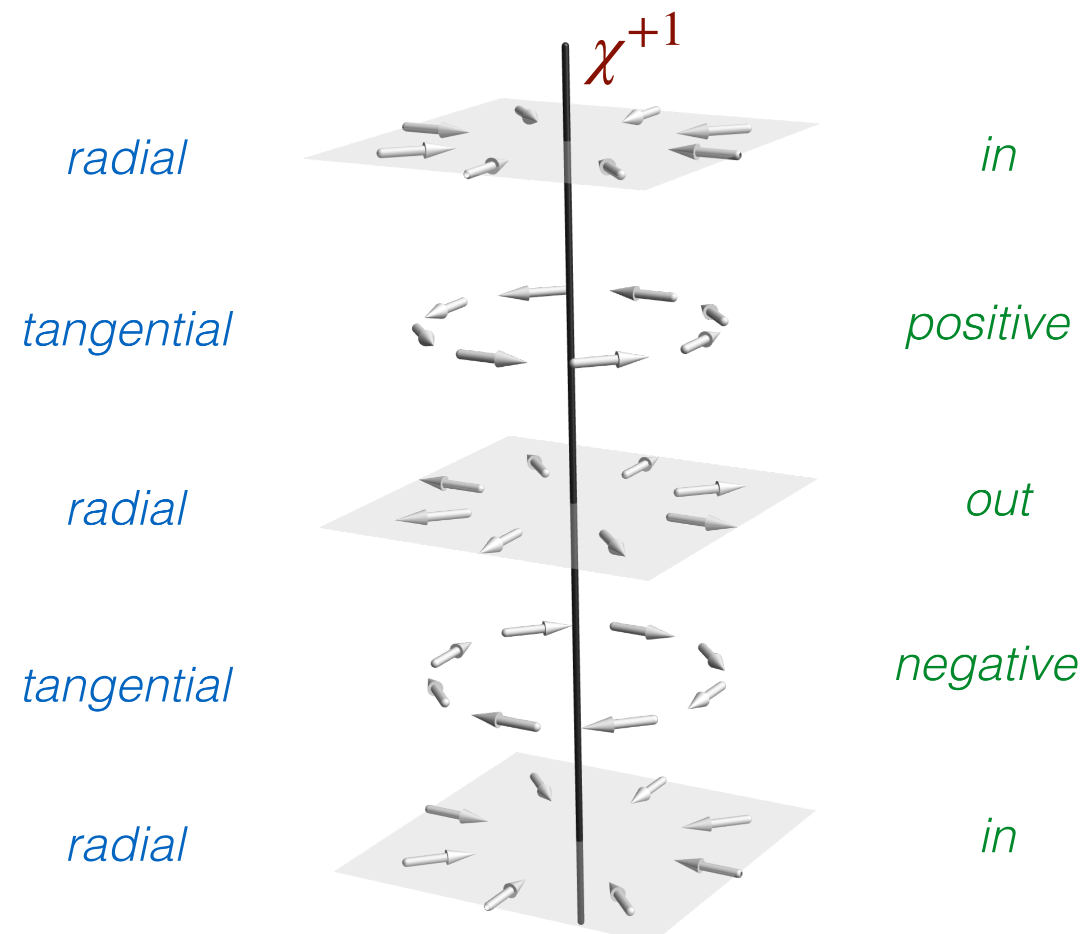
The issue is that the boundary conditions (for large r) vary along z, alternating between radial and tangential, and also in the directions of each.

V. Chiral Escape, Frustration and Defects

Consider places where the boundary condition is tangential ($q_0 z = \pi/2 + \pi \times$ integer), so either $\mathbf{n} = +\mathbf{e}_\phi$ or $\mathbf{n} = -\mathbf{e}_\phi$.

The key insight is that *escape up* is **left-handed** in the former case ($\mathbf{n} = +\mathbf{e}_\phi$) and **right-handed** in the latter ($\mathbf{n} = -\mathbf{e}_\phi$). The situation is exactly reversed for *escape down*.

Exercise: Check this!

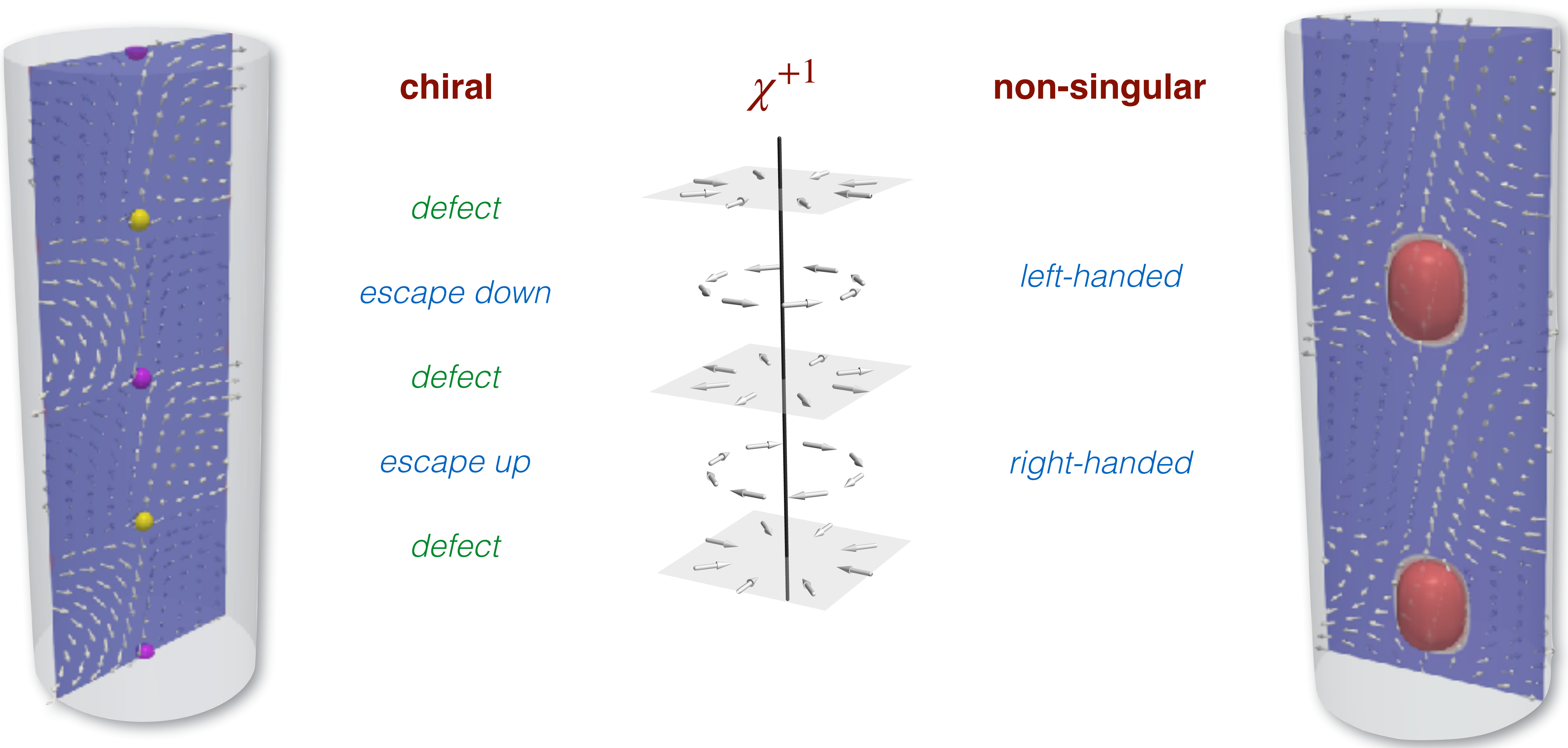


V. Chiral Escape, Frustration and Defects

The key insight is that *escape up* is **left-handed** in the former case ($\mathbf{n} = +\mathbf{e}_\phi$) and **right-handed** in the latter ($\mathbf{n} = -\mathbf{e}_\phi$). The situation is exactly reversed for *escape down*.

Thus, escape up everywhere is associated with *periodic reversals in the handedness*. These reversals are an amazing, and unique, type of **chiral topological soliton**.

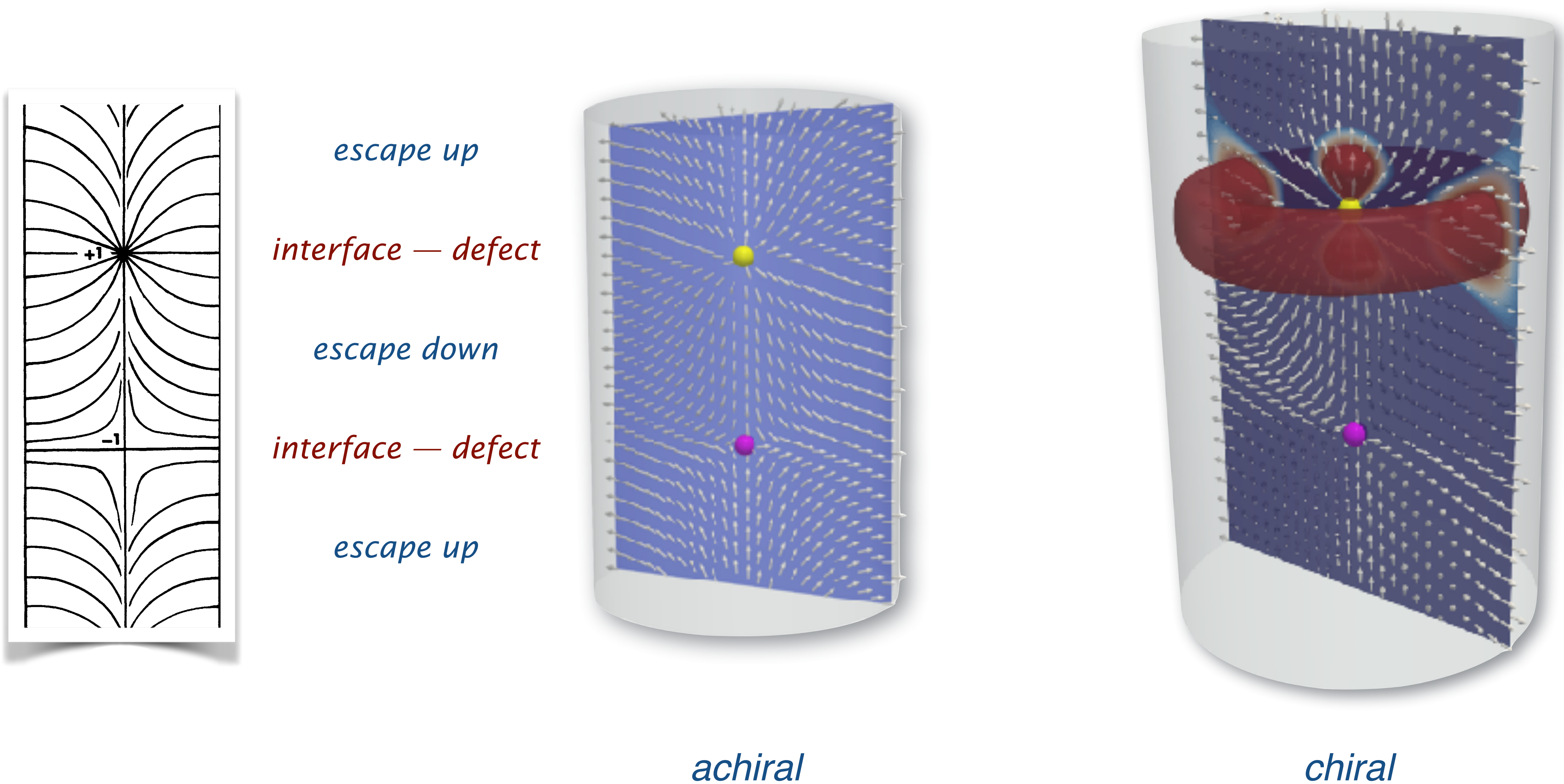
Maintaining a consistent handedness involves *periodic reversals in the direction of escape*. As escape up and escape down are topologically distinct, this creates a **string of point defects** along the axis of the escaped line.



V. Chiral Escape, Frustration and Defects

The same chiral topological solitons are seen in capillaries with purely normal anchoring.

Recall that the interface between escape up and escape down is marked by a point defect and that two different interfaces arise, depending on the order. Interfaces with a *degree* -1 point defect are **chiral**, while those with a *degree* $+1$ point defect **are not**.



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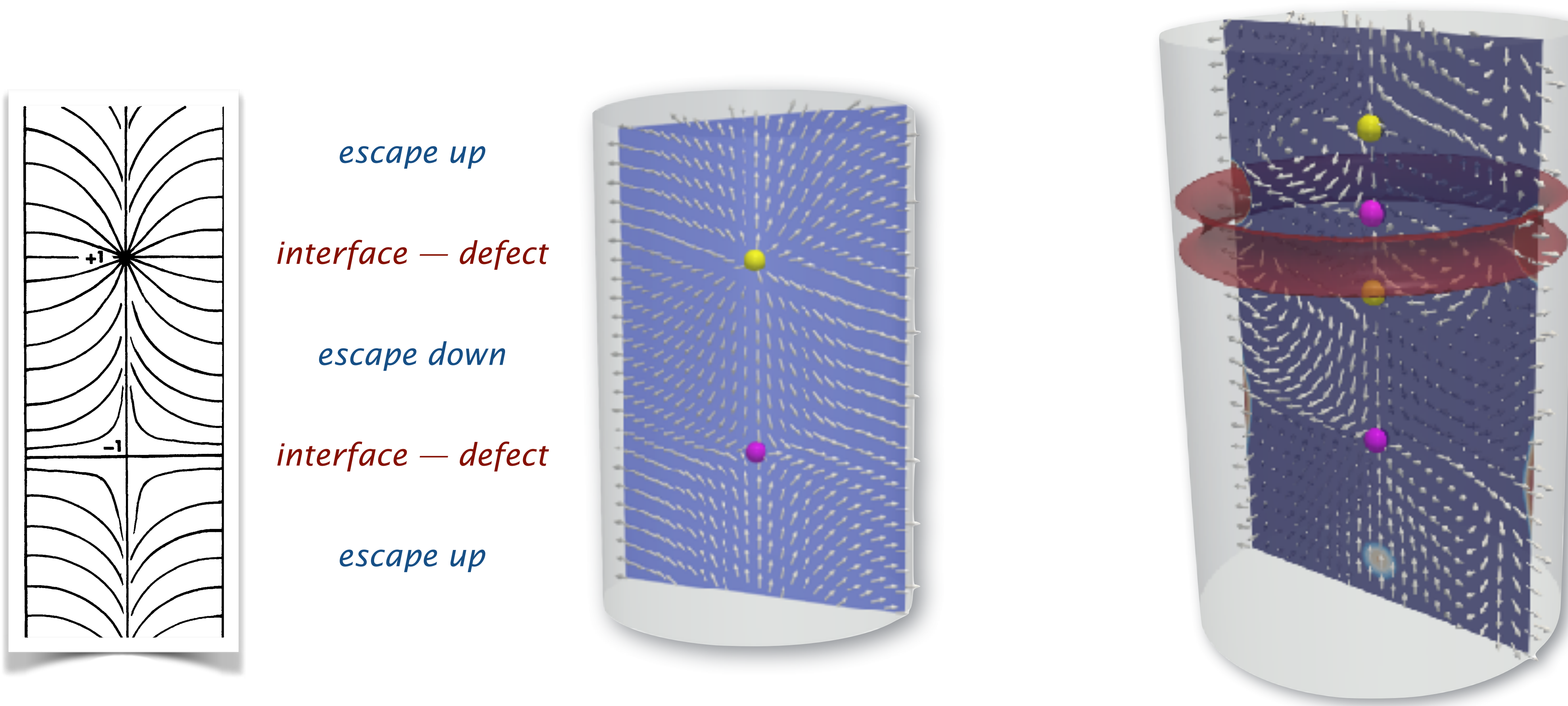
This is topological and cannot be removed by increasing the chirality.



achiral *chiral*

V. Chiral Escape, Frustration and Defects

To produce non-singular textures with a consistent handedness we need to go to higher winding numbers in the singular line.

First, observe that

$$n_x + in_y = \mathrm{e}^{i(q_0 z+\phi)} = \frac{(x+iy)\,\mathrm{e}^{iq_0 z}}{\sqrt{x^2+y^2}},$$

is the normalisation of $(x+iy)\,\mathrm{e}^{iq_0 z}$. A **+2 singular line** is the normalisation of $(x+iy)^2\,\mathrm{e}^{iq_0 z}$.

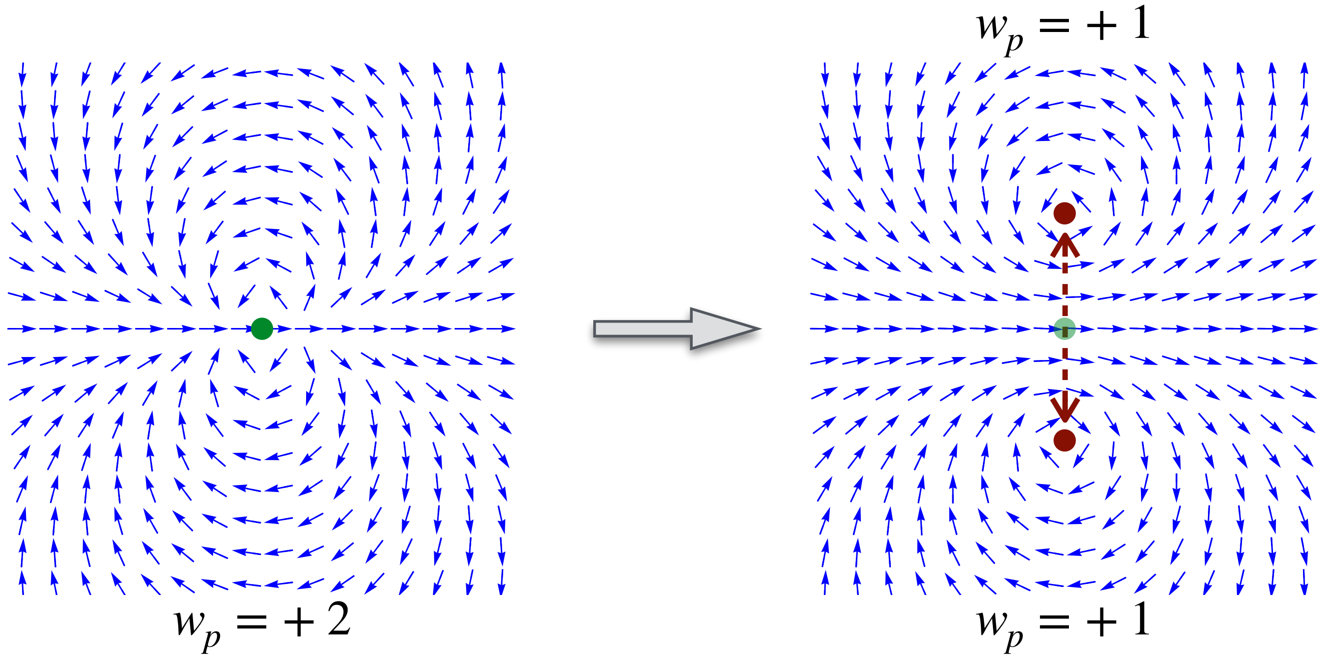
We use this unnormalised form to study its escape into the third dimension.

V. Chiral Escape, Frustration and Defects

A $+2$ singular line is the normalisation of $(x+iy)^2\, \mathrm{e}^{iq_0 z}$.

Consider its **unfolding** into two $+1$ lines at positions $\pm w(z)$ in the xy-plane

$$n_x + i n_y = \big(x + iy - w(z)\big)\big(x + iy + w(z)\big)\, \mathrm{e}^{iq_0 z}\,.$$



Near each of these, the local structure is $(x + iy = \pm\, w(z) + u + iv)$

$$n_x + i n_y \simeq \pm\,(u + iv)\, 2w(z)\, \mathrm{e}^{iq_0 z}\,.$$

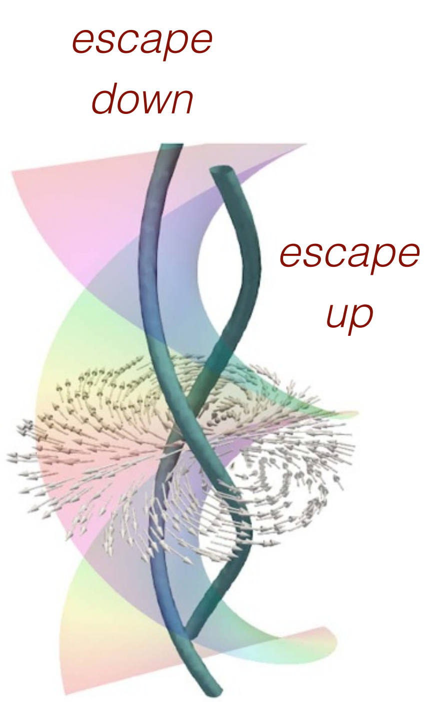
If we take $w(z) = iR\, \mathrm{e}^{-iq_0 z}$ these become $n_x + i n_y \simeq \pm\, 2R(iu - v)$ and each has a **consistent vortical profile** along the entire z-axis.

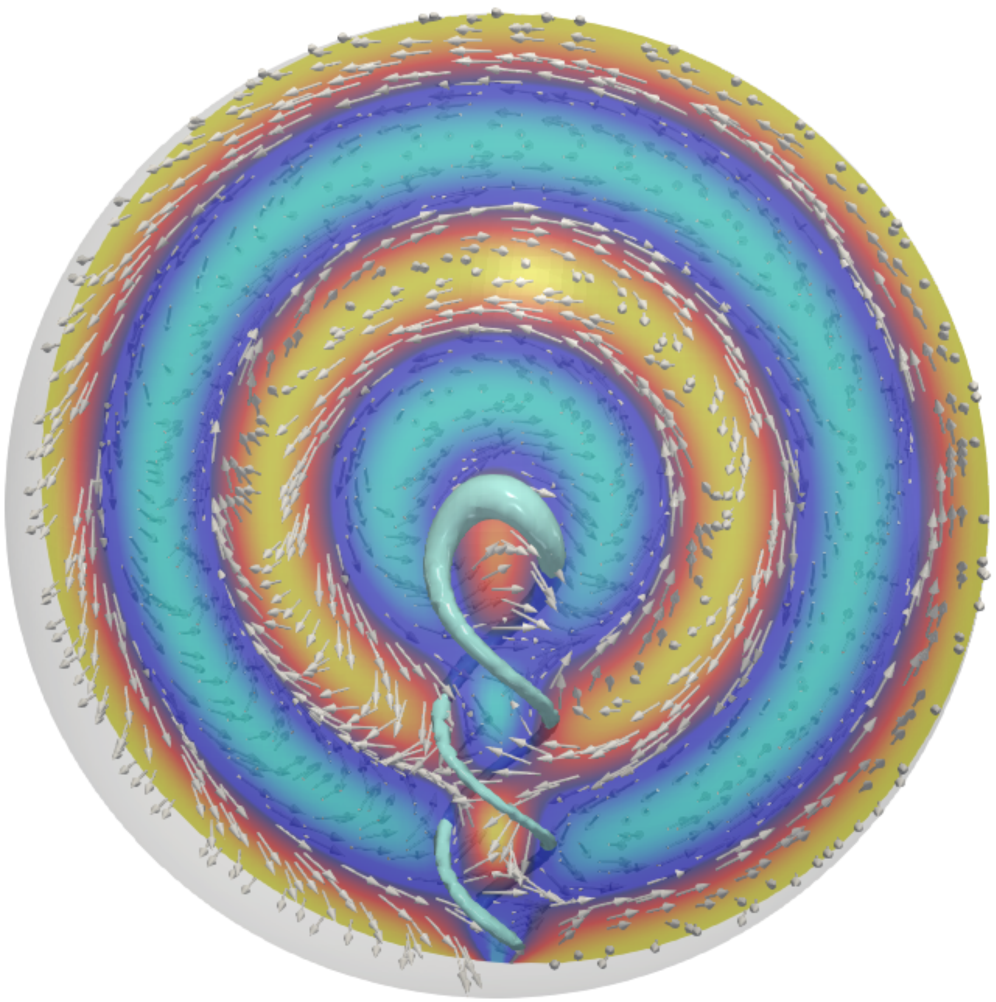
V. Chiral Escape, Frustration and Defects

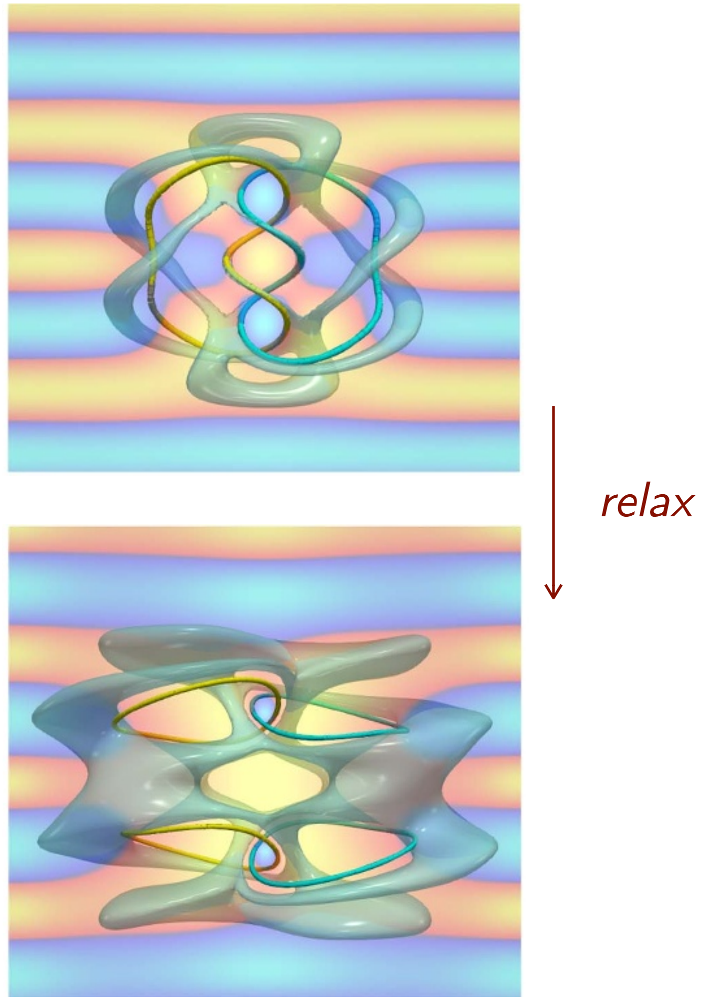
If we take $w(z) = iR\,\mathrm{e}^{-iq_0 z}$ these become $n_x + in_y \simeq \pm 2R(iu - v)$ and each has a **consistent vortical profile** along the entire z-axis.

We can then escape them **consistently**, the plus sign *escaping down* and the minus sign *escaping up*. The two escaped +1 lines wind around each other in a **left-handed helix**.

Exactly this motif is seen in textures of **cholesteric droplets** with tangential anchoring. If it is created locally (within a ball) in the cholesteric ground state, the system relaxes to a **heliknoton**.







heliknotons

Lecture 1: Textures in the Plane

Lecture 2: Escape from the Plane

Lecture 3: Hopfions and Chiral Topology

Lecture 4: Practicals — examples & discussion

I. The Hopf Fibration

II. Generation of Hopfions in $\mathbb{R}^3$

III. On Solid Angle

IV. Geometry of Director Fields

V. Cholesterics: Twist, Pitch Axis and Umbilic Lines

VI. Chiral Escape, Frustration and Defects