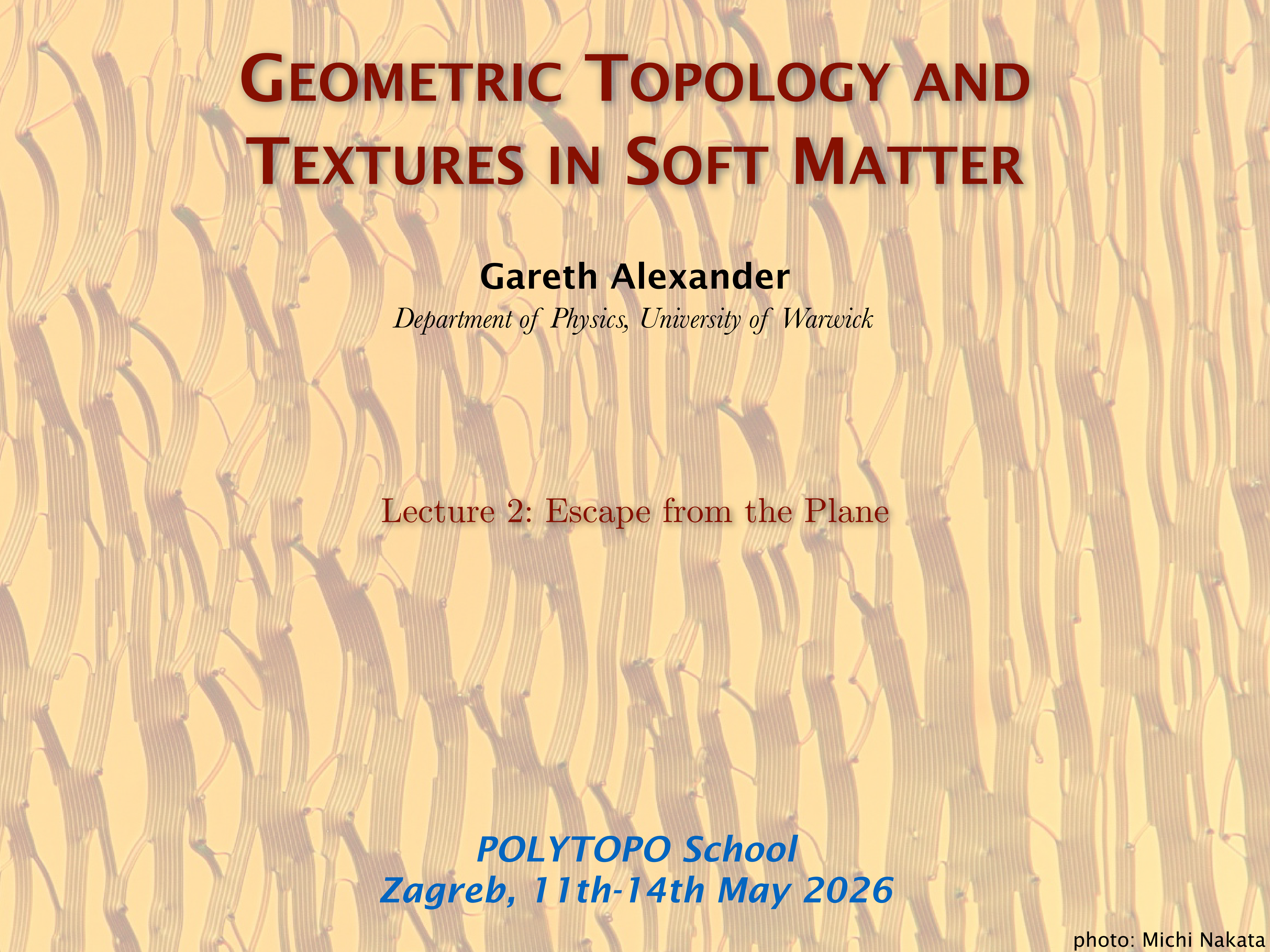

GEOMETRIC TOPOLOGY AND TEXTURES IN SOFT MATTER

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Lecture 2: Escape from the Plane

POLYTOPO School
Zagreb, 11th-14th May 2026

photo: Michi Nakata

Lecture 1: Textures in the Plane

Lecture 2: Escape from the Plane

Lecture 3: Hopfions and Chiral Topology

Lecture 4: Practicals — examples & discussion

I. Meyer's Escape into the Third Dimension

II. The Degree of Point Defects

III. Skyrmions

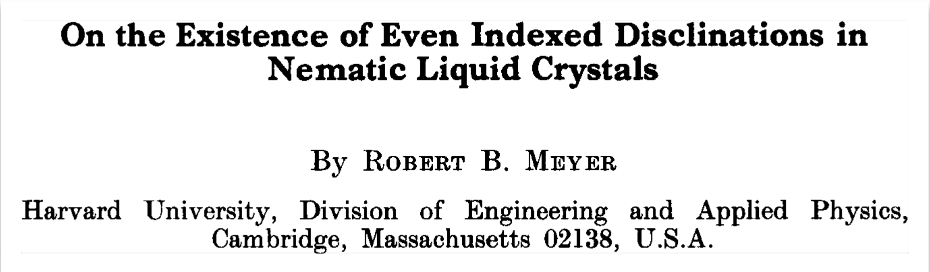
IV. Multi-skyrmion Textures and Energy Bound

I. Meyer's Escape into the Third Dimension

Robert Meyer (1973) wrote an influential paper on non-singular textures in cylindrical capillaries. A capillary of radius R has normal anchoring boundary conditions

$$\mathbf{n} = \cos\phi\, \mathbf{e}_x + \sin\phi\, \mathbf{e}_y = \mathbf{e}_r, \qquad \text{on} \quad r = R\,.$$

If this is continued throughout the capillary it will be singular (with winding number $+1$) along the axis. As Meyer described, this can be avoided by allowing the director to tilt out of the horizontal plane and point vertically, either up or down, along the axis. This has become known as '**escape into the third dimension**'.

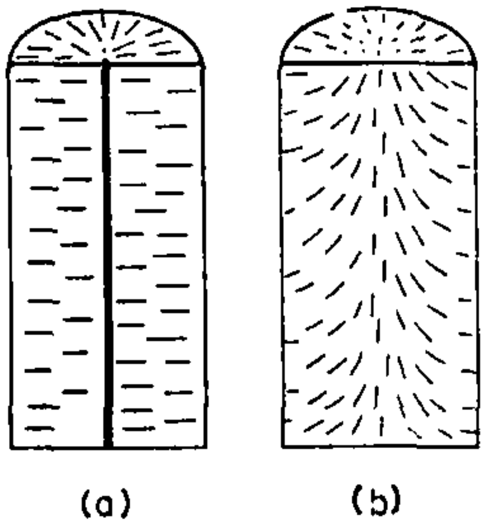


On the Existence of Even Indexed Disclinations in Nematic Liquid Crystals

By Robert B. Meyer

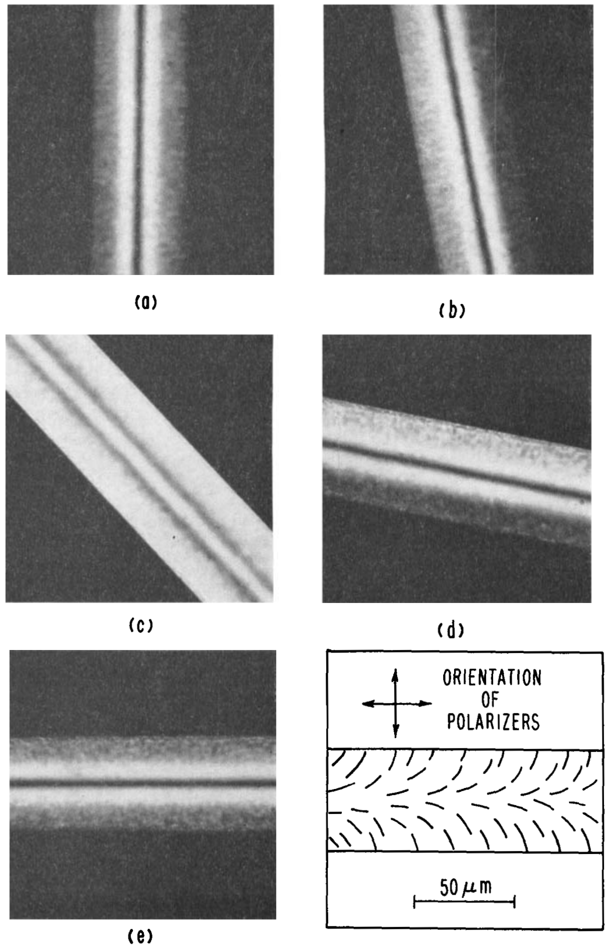
Harvard University, Division of Engineering and Applied Physics,
Cambridge, Massachusetts 02138, U.S.A.

Fig. 1



(a) (b)

Fig. 9

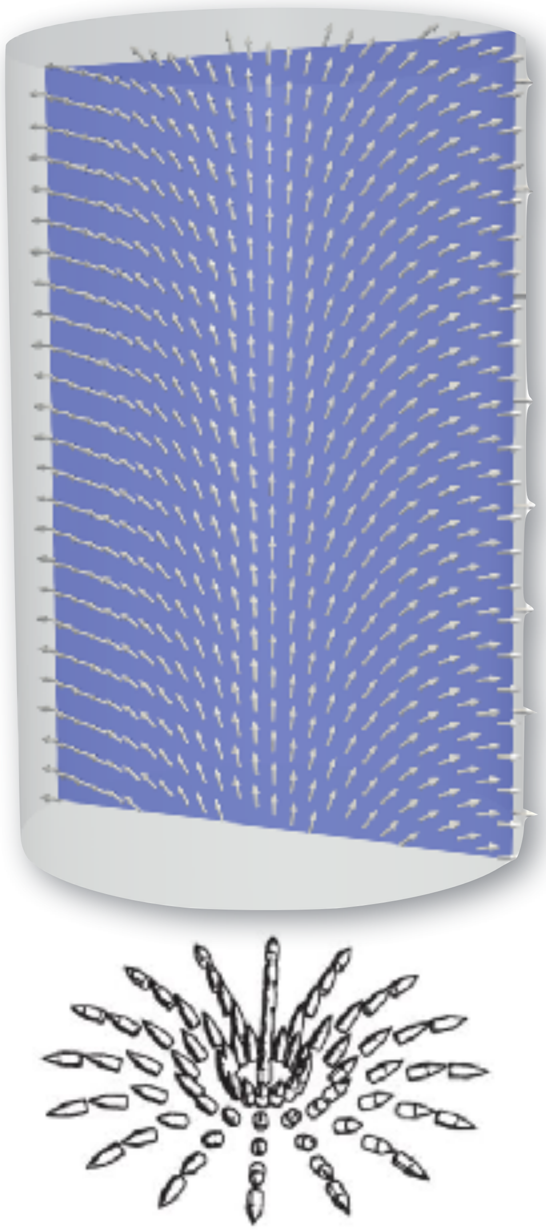


A cylindrical sample rotated counterclockwise ($a \to e$) between crossed linear polarizers, with a sketch of the continuous structure consistent with the extinction pattern.

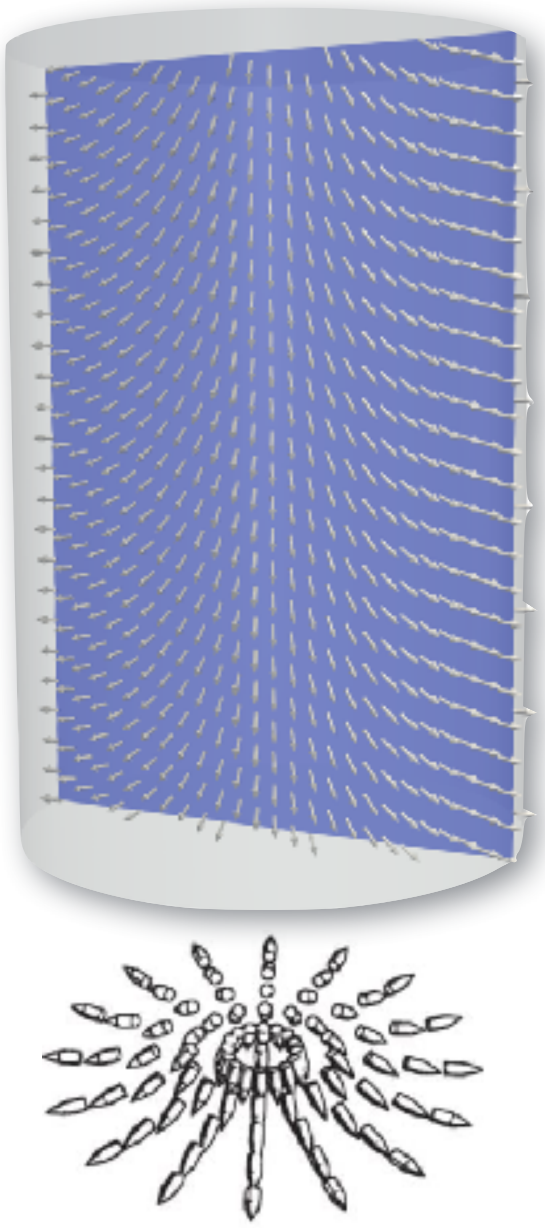
I. Meyer’s Escape into the Third Dimension

This has become known as ‘**escape into the third dimension**’.

It can be described by a director field $\mathbf{n} = \cos\theta\,\mathbf{e}_z + \sin\theta\,\mathbf{e}_r$, where θ takes the value $\pi/2$ at all points of the boundary ($r = R$) and either the value 0 (‘**escape up**’) or the value π (‘**escape down**’) at $r = 0$.



escape up



escape down

I. Meyer's Escape into the Third Dimension

It is interesting to determine the form of the tilt angle θ in the escape textures explicitly.

Exercise: Determine the tilt angle $\theta(r)$ that minimises the one-elastic-constant free energy

$$F = \int \frac{K}{2} |\nabla \mathbf{n}|^2 \, dV = \frac{K}{2} \int \big(\partial_r \theta\big)^2 + \frac{\sin^2\theta}{r^2} \, dV \, .$$

assuming that it depends only on the radial distance r from the capillary axis.

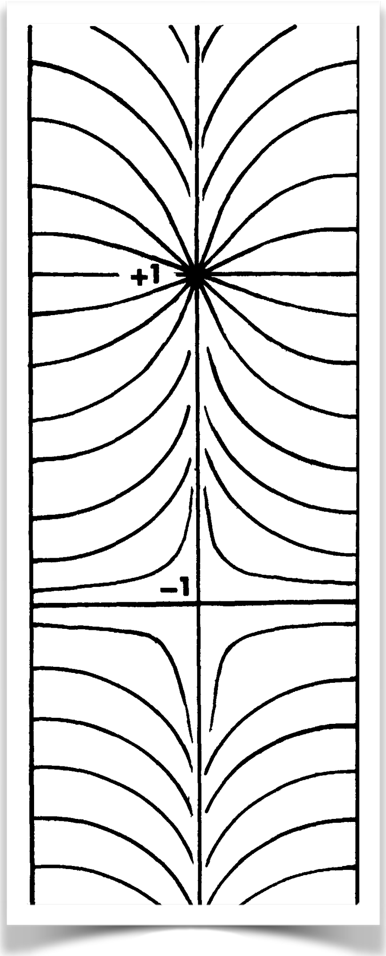
Hint: Use the 'BPS' method from the end of this lecture.

Compare with the energy of the unescaped singular line. Find the solution also for the case of tangential boundary conditions, $\mathbf{n} = \mathbf{e}_\phi$ on $r = R$.

I. Meyer’s Escape into the Third Dimension

Escape up and escape down are **inequivalent** — they are not **homotopic**.

A way to see this is to realise both textures in the same capillary and look at the interface between them. The interface is marked by a singularity in the director field at a single point, illustrating that there is no continuous interpolation.



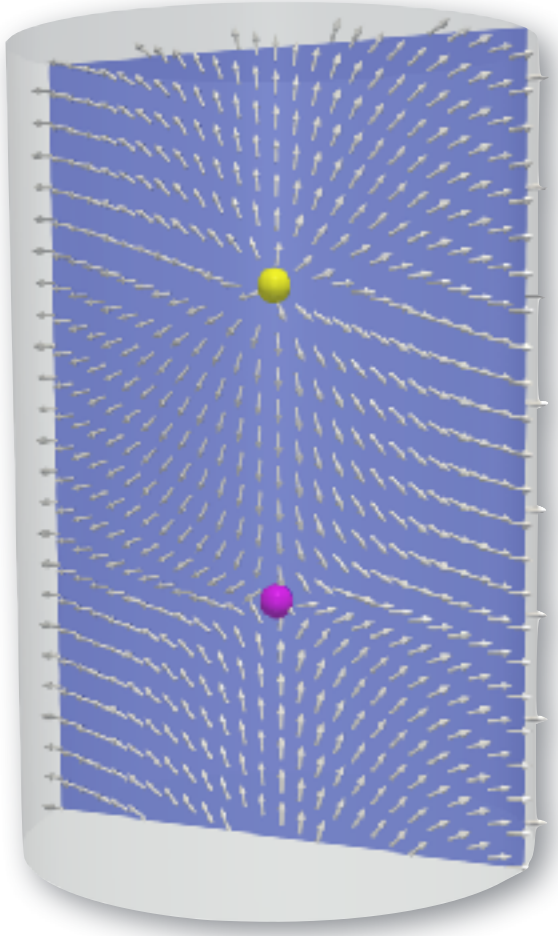
escape up

interface — defect

escape down

interface — defect

escape up

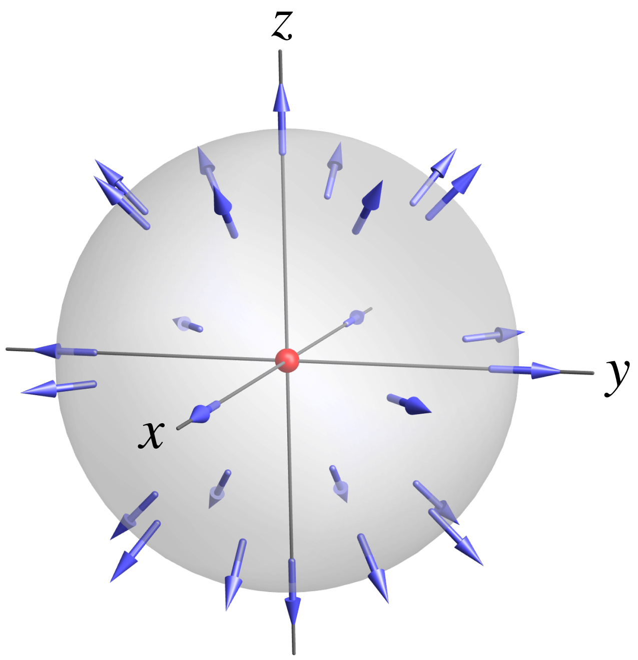


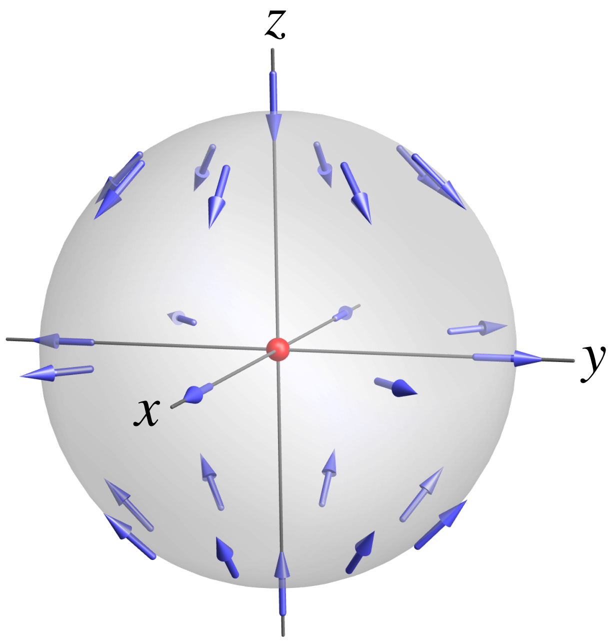
These singularities are examples of three-dimensional **point defects**.

II. The Degree of Point Defects

Local models for the defects at interfaces between escape up and escape down are given by

$$\mathbf{n} = \frac{x\,\mathbf{e}_x + y\,\mathbf{e}_y + z\,\mathbf{e}_z}{\sqrt{x^2+y^2+z^2}}, \qquad \text{and} \qquad \mathbf{n} = \frac{x\,\mathbf{e}_x + y\,\mathbf{e}_y - z\,\mathbf{e}_z}{\sqrt{x^2+y^2+z^2}}.$$





These three-dimensional point defects have properties analogous to their two-dimensional counterparts.

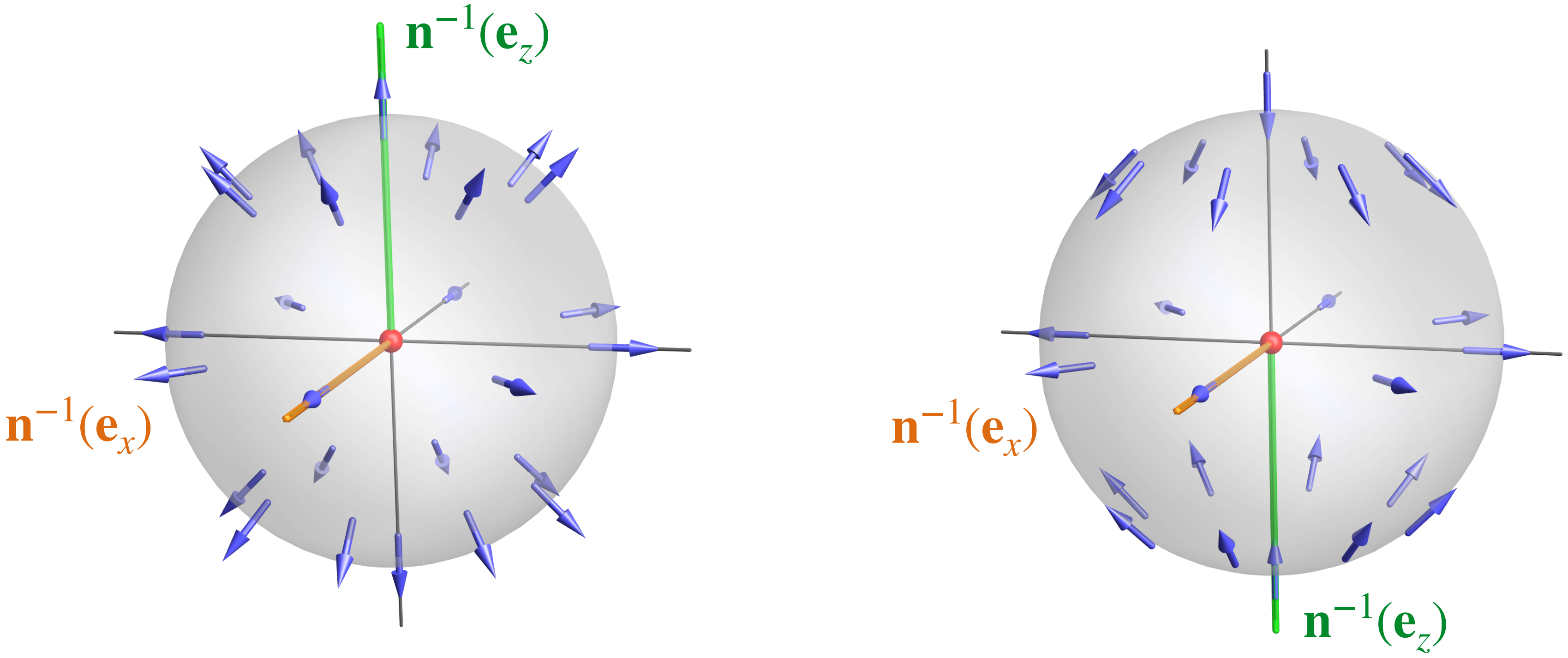
They have an **integer topological invariant** analogous to the winding number. This integer, called the **degree**, counts the number of times the director around the defect covers all possible orientations on the unit sphere.

It can be computed using the Pontrjagin-Thom construction, or by an integral.

II. The Degree of Point Defects

The set of all orientations in three-dimensional space forms the **unit sphere**, S^2. Fix a particular orientation, say (θ, ϕ) in spherical polar coordinates.

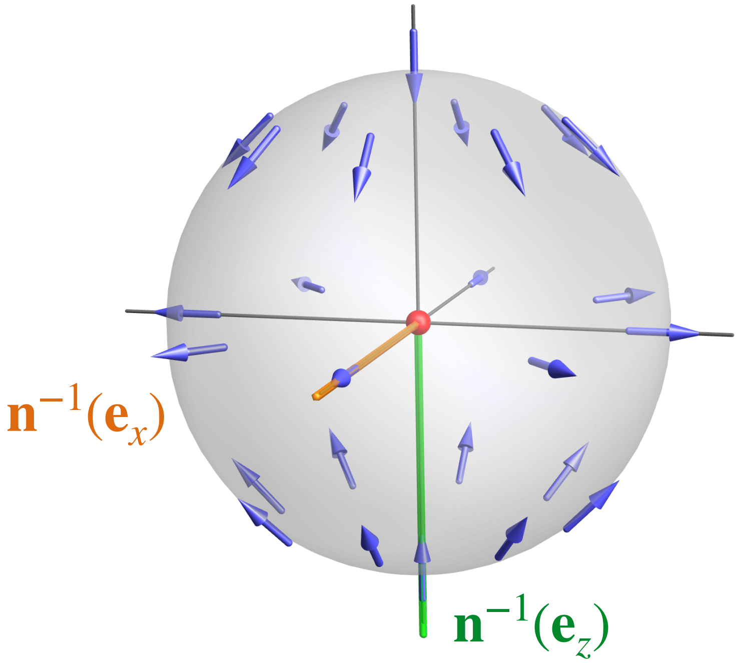
The **inverse image** $\mathbf{n}^{-1}(\theta, \phi)$ is the set of all points in the material where the director has that orientation. This is 'nice' for almost every choice of (θ, ϕ), called a **regular value**.



By continuity, nearby orientations have nearby inverse images; how they are arranged gives a **framing** of $\mathbf{n}^{-1}(\theta, \phi)$. This can be used to orient the inverse image. Following nearby orientations on the sphere that circulate around (θ, ϕ) positively, their inverse images will similarly circulate around $\mathbf{n}^{-1}(\theta, \phi)$ and this provides an orientation by the **right-hand rule**.

II. The Degree of Point Defects

If Σ is an oriented closed surface the inverse image $\mathbf{n}^{-1}(\theta,\phi)$ will intersect it in a set of isolated points. Each intersection acquires a sign according to whether the orientation of the inverse image aligns with, or against, that of the surface.



This signed count is the **degree** of the map $\mathbf{n} : \Sigma \to S^2$, or the **charge** Q of the point defect

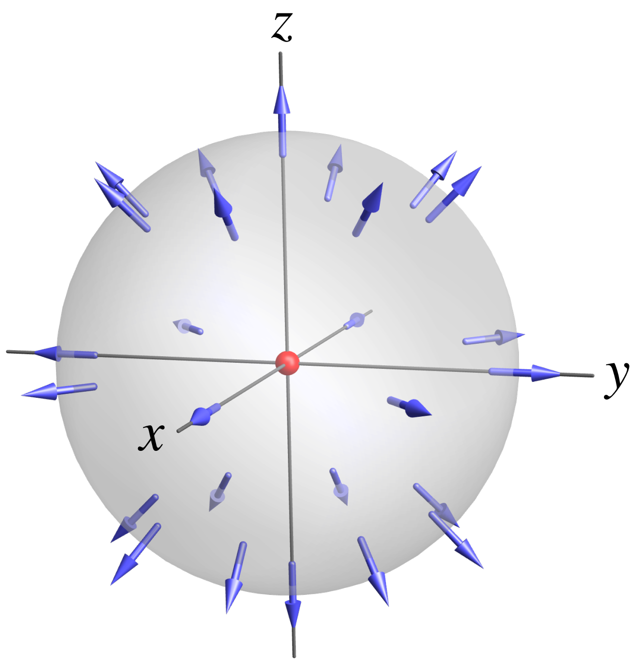
$$\deg_{\mathbf{n}}(\Sigma) = \mathrm{Int}\big(\Sigma, \mathbf{n}^{-1}(\theta,\phi)\big) = Q\,.$$

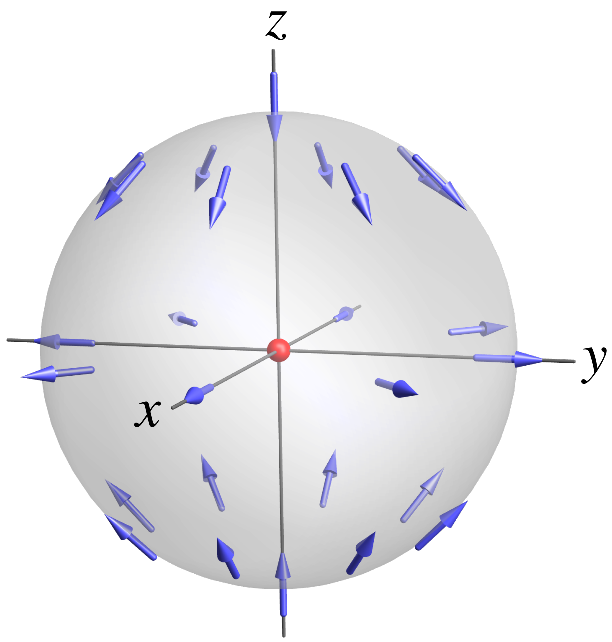
It can be any integer. It is a part of the Pontrjagin-Thom theorem that it identifies the homotopy class of the texture.

II. The Degree of Point Defects

Exercise: Use the Pontrjagin-Thom construction to determine the degrees of the two model point defects

$$\mathbf{n} = \frac{x\,\mathbf{e}_x + y\,\mathbf{e}_y + z\,\mathbf{e}_z}{\sqrt{x^2+y^2+z^2}}, \qquad \text{and} \qquad \mathbf{n} = \frac{x\,\mathbf{e}_x + y\,\mathbf{e}_y - z\,\mathbf{e}_z}{\sqrt{x^2+y^2+z^2}}\,.$$





II. The Degree of Point Defects

We now describe how the degree can be computed from a suitable integral, using the differential properties of the texture.

Write the director as

$$\mathbf{n} = \sin\theta\left[\cos\phi\,\mathbf{e}_x + \sin\phi\,\mathbf{e}_y\right] + \cos\theta\,\mathbf{e}_z,$$

with (θ, ϕ) spherical polar coordinates for the point on S^2 corresponding to the orientation $\mathbf{n}$. $\theta = \theta(\mathbf{x}), \phi = \phi(\mathbf{x})$ are also functions of position in the texture.

II. The Degree of Point Defects

Write the director as $\mathbf{n} = \sin\theta \left[\cos\phi\,\mathbf{e}_x + \sin\phi\,\mathbf{e}_y\right] + \cos\theta\,\mathbf{e}_z$.

We consider the differential form

$$d\mathbf{n} = d\theta \otimes \left[\cos\theta\left(\cos\phi\,\mathbf{e}_x + \sin\phi\,\mathbf{e}_y\right) - \sin\theta\,\mathbf{e}_z\right] + \sin\theta\,d\phi \otimes \left[-\sin\phi\,\mathbf{e}_x + \cos\phi\,\mathbf{e}_y\right].$$

Take an antisymmetric product on both parts — a *wedge product* on the differential forms and a *cross product* on the vectors[†]

$$d\mathbf{n} \times d\mathbf{n} = d\theta \wedge \sin\theta\,d\phi \otimes \mathbf{n} + \sin\theta\,d\phi \wedge d\theta \otimes \left(-\mathbf{n}\right).$$

We therefore find

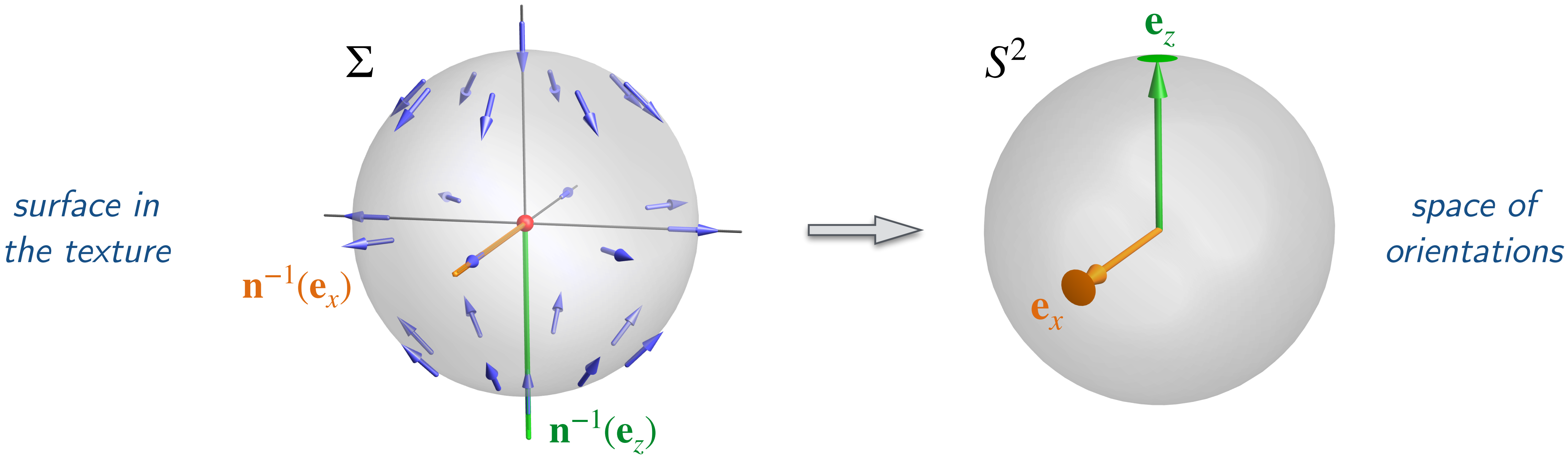
$$\mathbf{n} \cdot d\mathbf{n} \times d\mathbf{n} = 2\sin\theta\,d\theta \wedge d\phi\,.$$

[†] I do not know of an official notation for this but have chosen to write it just using the cross product; the context should make clear what is meant.

II. The Degree of Point Defects

$$\mathbf{n}\cdot d\mathbf{n}\times d\mathbf{n} = 2\sin\theta\, d\theta\wedge d\phi\,.$$

For any small patch of a surface Σ, the variation of the director on that patch sweeps out some area of orientations on the unit sphere and the differential form $\mathbf{n}\cdot d\mathbf{n}\times d\mathbf{n}$ measures twice that area.



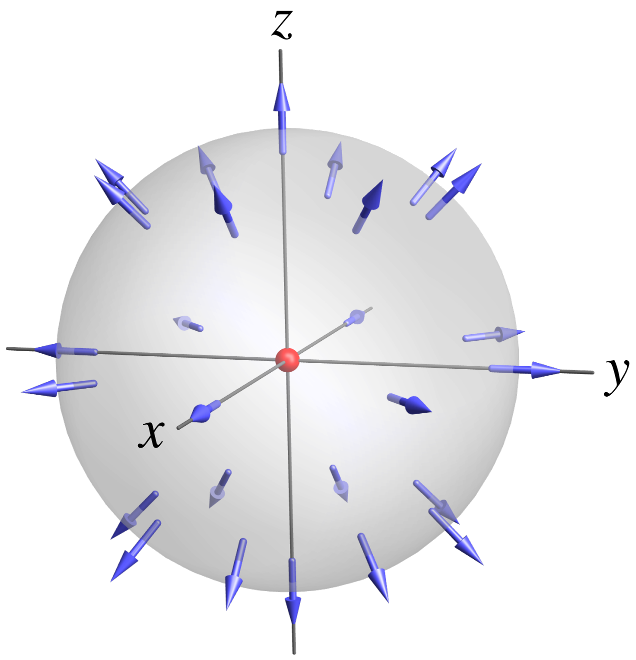
It follows that the integral over any closed surface will be equal to twice the total area covered on S^2. Hence, we have

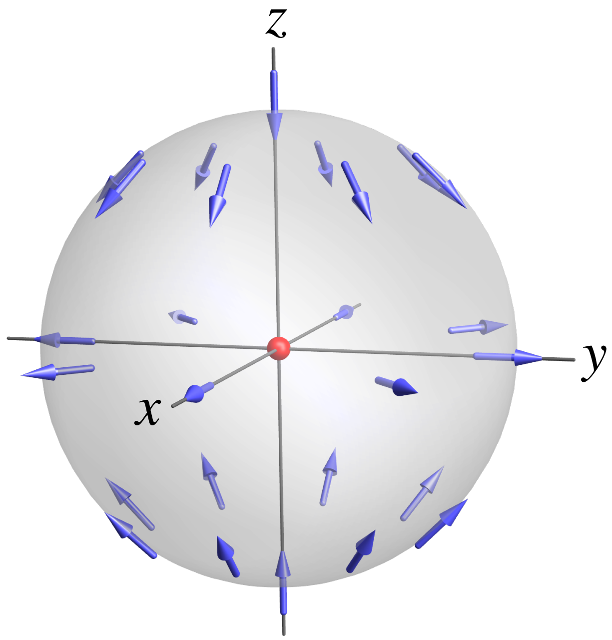
$$\int_\Sigma \mathbf{n}\cdot d\mathbf{n}\times d\mathbf{n} = 2\int_\Sigma \sin\theta\, d\theta\wedge d\phi = 2\times 4\pi\,\deg_{\mathbf{n}}(\Sigma) = 8\pi Q\,.$$

II. The Degree of Point Defects

Exercise: Use the integral formula to determine the degrees of the two model point defects

$$\mathbf{n} = \frac{x\,\mathbf{e}_x + y\,\mathbf{e}_y + z\,\mathbf{e}_z}{\sqrt{x^2 + y^2 + z^2}}, \qquad \text{and} \qquad \mathbf{n} = \frac{x\,\mathbf{e}_x + y\,\mathbf{e}_y - z\,\mathbf{e}_z}{\sqrt{x^2 + y^2 + z^2}}.$$





II. The Degree of Point Defects

$$\deg_{\mathbf{n}}(\Sigma) = \frac{1}{8\pi} \int_{\Sigma} \mathbf{n} \cdot d\mathbf{n} \times d\mathbf{n} \,.$$

When Σ does not enclose any defects the integral is **zero**. For two-dimensional defects we showed the analogous result using Green's theorem in the plane. Here, the corresponding calculation would use the **divergence theorem**.

Working in the framework of the exterior calculus becomes increasingly common. In that framework it amounts to showing that $\mathbf{n} \cdot d\mathbf{n} \times d\mathbf{n}$ is a **closed form**[†]. Calculating explicitly, using $d^2 = 0$ and disentangling the differential form and vector parts,

$$d\big(\mathbf{n} \cdot d\mathbf{n} \times d\mathbf{n}\big) = d\mathbf{n} \cdot d\mathbf{n} \times d\mathbf{n} = 6\big(\partial_x \mathbf{n} \cdot \partial_y \mathbf{n} \times \partial_z \mathbf{n}\big)\, dx \wedge dy \wedge dz \,.$$

As $\mathbf{n}$ is a unit vector, the derivatives $\partial_a \mathbf{n}$ all lie in the plane perpendicular to $\mathbf{n}$ and the triple scalar product vanishes because it involves three coplanar vectors.

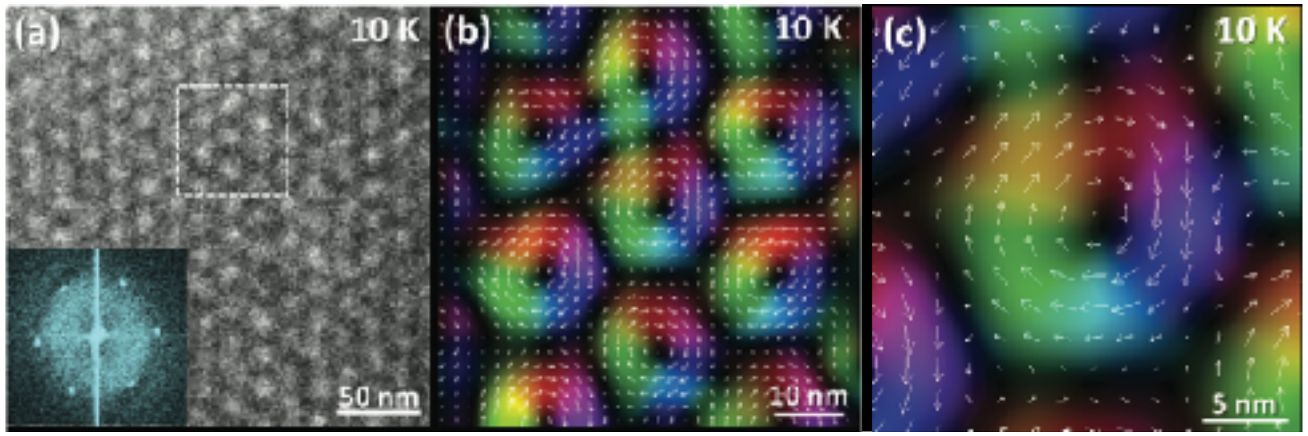
Hence, $\mathbf{n} \cdot d\mathbf{n} \times d\mathbf{n}$ is a closed form and represents a **cohomology class** in $H^2(\mathbb{R}^3 \backslash \mathscr{D})$.

[†] This is done using a structural result, namely that d commutes with pull back; here, I give a calculation that does not need this.

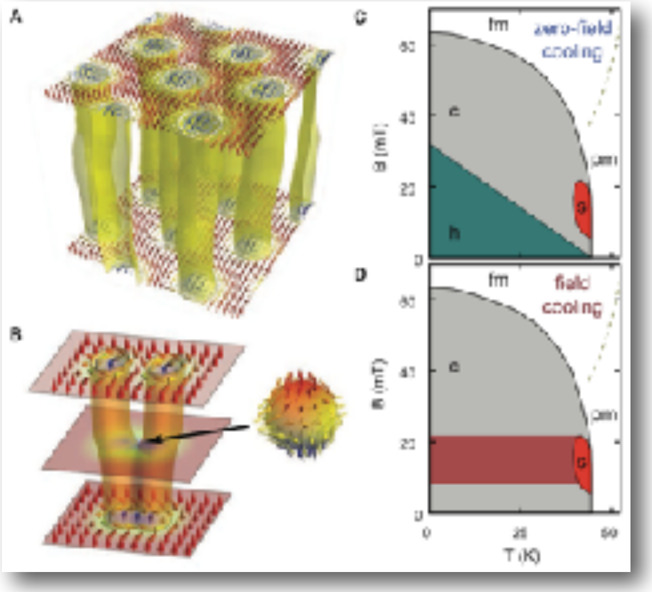
III. Skyrmions

Often described as ‘**magnetic whorls**’, skyrmions are textures with three-dimensional orientation on a two-dimensional (flat) plane†.

magnetic skyrmions

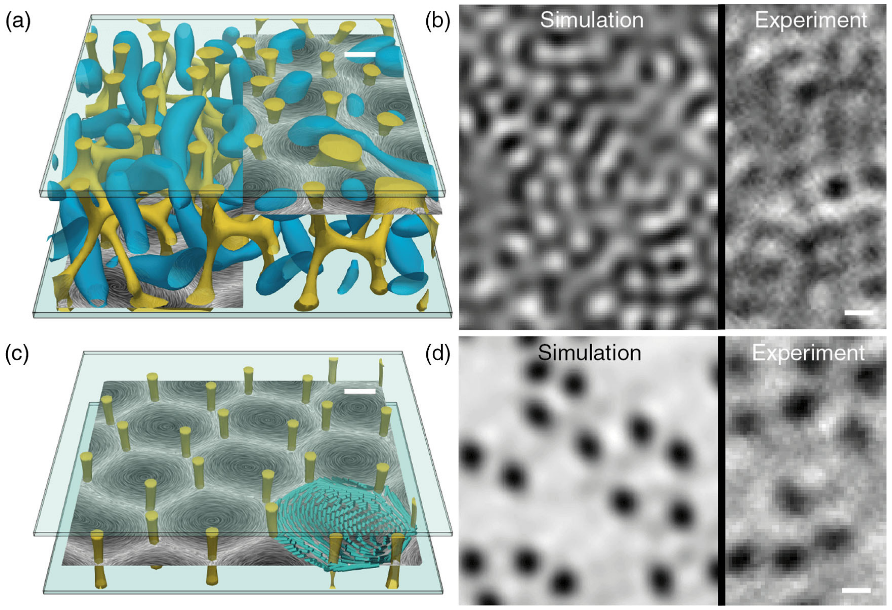


Yu *et al*. Nature **465**, 901 (2010)

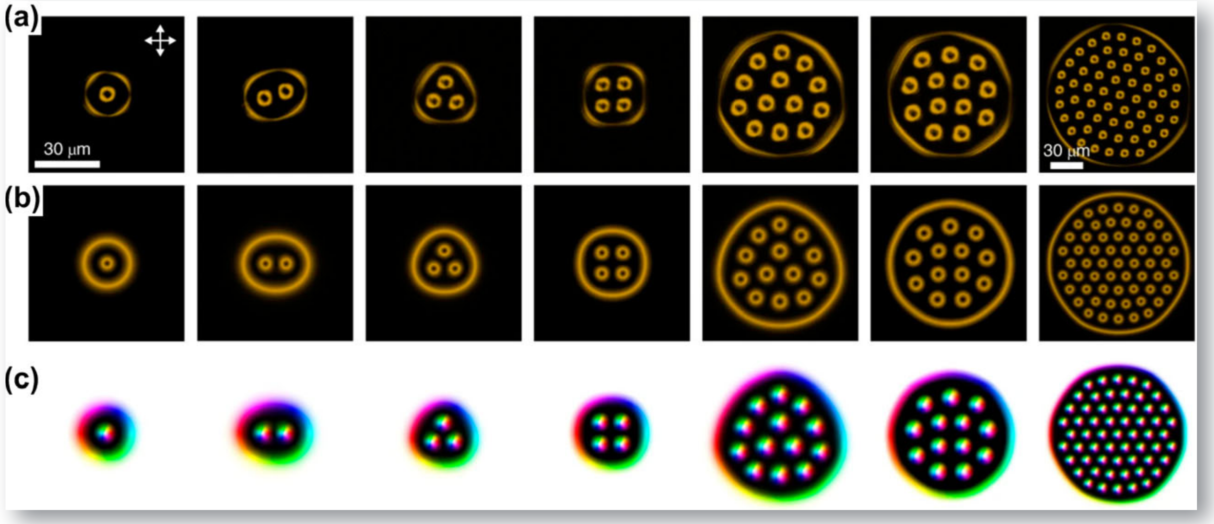


Milde *et al*. Science **340**, 1076 (2013)

liquid crystal skyrmions



Pišljar *et al*., Phys. Rev. X **6**, 118 (2022)



Foster *et al*., Nat. Phys. **15**, 655 (2019)

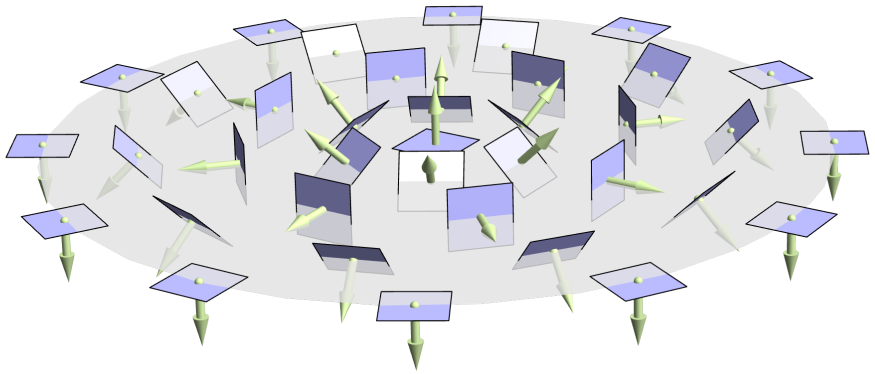
†If there are three spatial dimensions the texture only varies in two of them.

III. Skyrmions

A basic example is provided by the director field

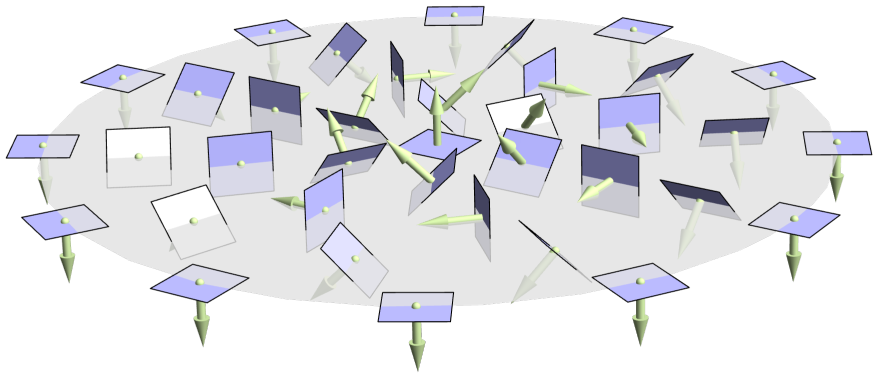
$$\mathbf{n} = \cos\frac{\pi r}{R}\,\mathbf{e}_z + \sin\frac{\pi r}{R}\left[\cos(\phi-\alpha)\,\mathbf{e}_x + \sin(\phi-\alpha)\,\mathbf{e}_y\right],$$

on the planar disc $r \leq R$. This is a **one-parameter family** of skyrmion textures, depending on the parameter α. When $\alpha = 0$ the texture has a radial appearance and is called a **Néel skyrmion**; when $\alpha = \pi/2$ it is vortical and called a **Bloch skyrmion**.



Néel skyrmion

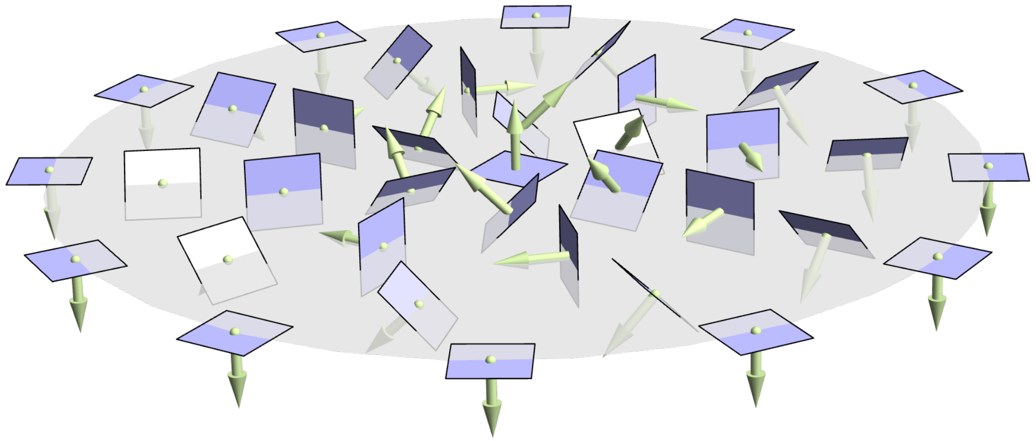
$(\alpha = 0)$



Bloch skyrmion

$(\alpha = \pi/2)$

III. Skyrmions



$$\mathbf{n} = \cos\frac{\pi r}{R}\,\mathbf{e}_z + \sin\frac{\pi r}{R}\left[\cos(\phi-\alpha)\,\mathbf{e}_x + \sin(\phi-\alpha)\,\mathbf{e}_y\right].$$

The director covers all orientations on the unit sphere once and has **degree 1**. This can be analysed by the Pontrjagin-Thom construction, say by looking at the inverse image of the orientation $(\pi/2,0)$, which is the point $r = R/2$, $\phi = \alpha$ of the disc.

Is the intersection number of this inverse image with the disc positive or negative?

We interpret it as the **number of skyrmions** on the disc; it is well-defined as long as we maintain the boundary condition $\mathbf{n} = -\mathbf{e}_z$ at the edge of the disc.

III. Skyrmions

$$\mathbf{n} = \cos\frac{\pi r}{R}\,\mathbf{e}_z + \sin\frac{\pi r}{R}\left[\cos(\phi-\alpha)\,\mathbf{e}_x + \sin(\phi-\alpha)\,\mathbf{e}_y\right],$$

To illustrate methodology, we compute the degree also using the integral formula.

Variation of the direction gives

$$d\mathbf{n} = \frac{\pi}{R}\,dr \otimes \left[\cos\frac{\pi r}{R}\left(\cos(\phi-\alpha)\,\mathbf{e}_x + \sin(\phi-\alpha)\,\mathbf{e}_y\right) - \sin\frac{\pi r}{R}\,\mathbf{e}_z\right] + \sin\frac{\pi r}{R}\,d\phi \otimes \left[\cos(\phi-\alpha)\,\mathbf{e}_y - \sin(\phi-\alpha)\,\mathbf{e}_x\right].$$

We find

$$\mathbf{n}\cdot d\mathbf{n}\times d\mathbf{n} = \frac{2\pi}{R}\sin\frac{\pi r}{R}\,dr\wedge d\phi = \frac{2\pi^2}{R^2}\frac{\sin \pi r/R}{\pi r/R}\,dA\,.$$

Integrating over the entire disc we obtain

$$\int_D \mathbf{n}\cdot d\mathbf{n}\times d\mathbf{n} = \int_{\phi=0}^{2\pi}\int_{r=0}^{R}\frac{2\pi}{R}\sin\frac{\pi r}{R}\,dr\,d\phi = 4\pi\left[-\cos\frac{\pi r}{R}\right]_{r=0}^{R} = 8\pi\,.$$

The skyrmion number is 1; there is a single skyrmion on the disc.

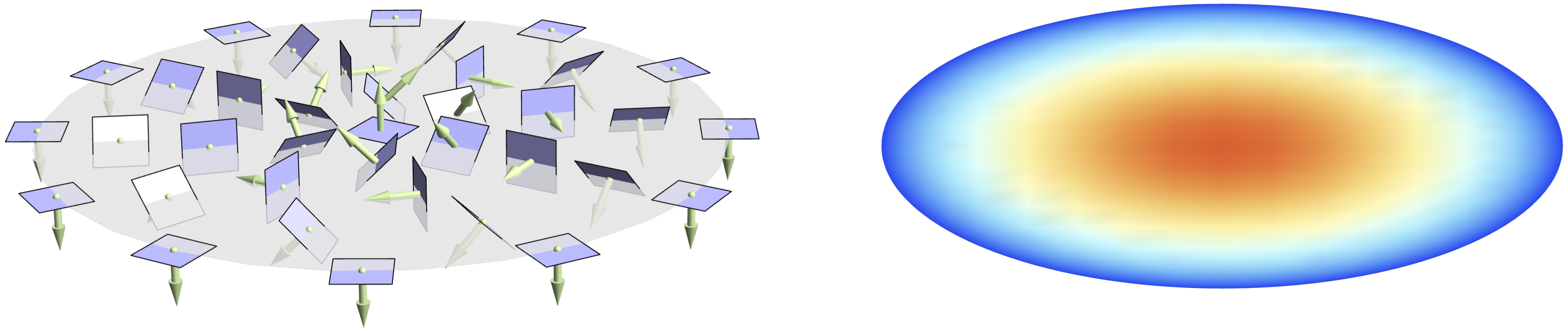
III. Skyrmions

$$\mathbf{n} \cdot d\mathbf{n} \times d\mathbf{n} = \frac{2\pi}{R} \sin \frac{\pi r}{R}\, dr \wedge d\phi = \frac{2\pi^2}{R^2} \frac{\sin \pi r/R}{\pi r/R}\, dA\,.$$

The function

$$\frac{\pi}{4R^2} \frac{\sin \pi r/R}{\pi r/R},$$

acts as a **skyrmion density**; integrating over the disc returns the number of skyrmions. The location of the maximum of this function — at the centre of the disc — serves as a proxy for the **position** of the skyrmion.

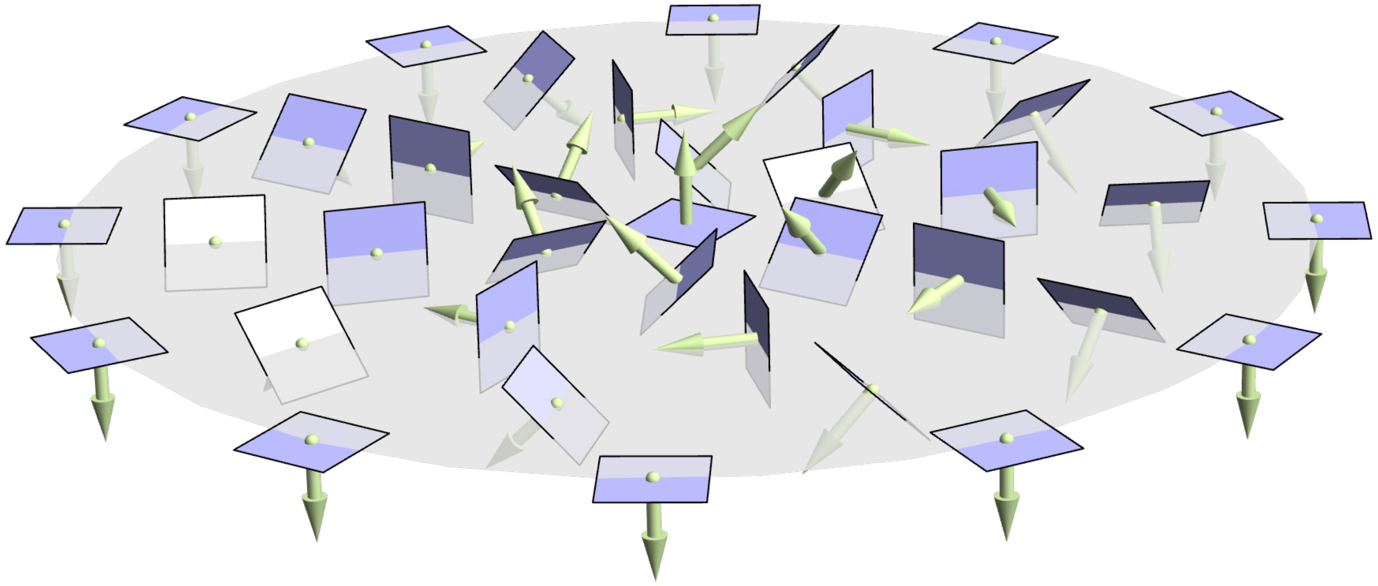


III. Skyrmions

Recall that $\mathbf{n} \cdot d\mathbf{n} \times d\mathbf{n} = 2 \sin\theta \, d\theta \wedge d\phi$. Thus we can say

$$\text{spherical area} = \sin\theta \, d\theta \wedge d\phi = \frac{\pi^2}{R^2} \frac{\sin \pi r/R}{\pi r/R} \, dA = \text{`curvature'} \times \text{area on the disc}$$

This is intended to echo the **Gaussian curvature** of surfaces. The skyrmion density is a type of **curvature** — it is the *curvature of the vector bundle of planes perpendicular to the director field*.



III. Skyrmions

From the differential $d\mathbf{n}$ we see that the vectors

$$\mathbf{e}_1 = \cos\frac{\pi r}{R}\left[\cos(\phi-\alpha)\,\mathbf{e}_x + \sin(\phi-\alpha)\,\mathbf{e}_y\right] - \sin\frac{\pi r}{R}\,\mathbf{e}_z, \qquad \mathbf{e}_2 = \cos(\phi-\alpha)\,\mathbf{e}_y - \sin(\phi-\alpha)\,\mathbf{e}_x,$$

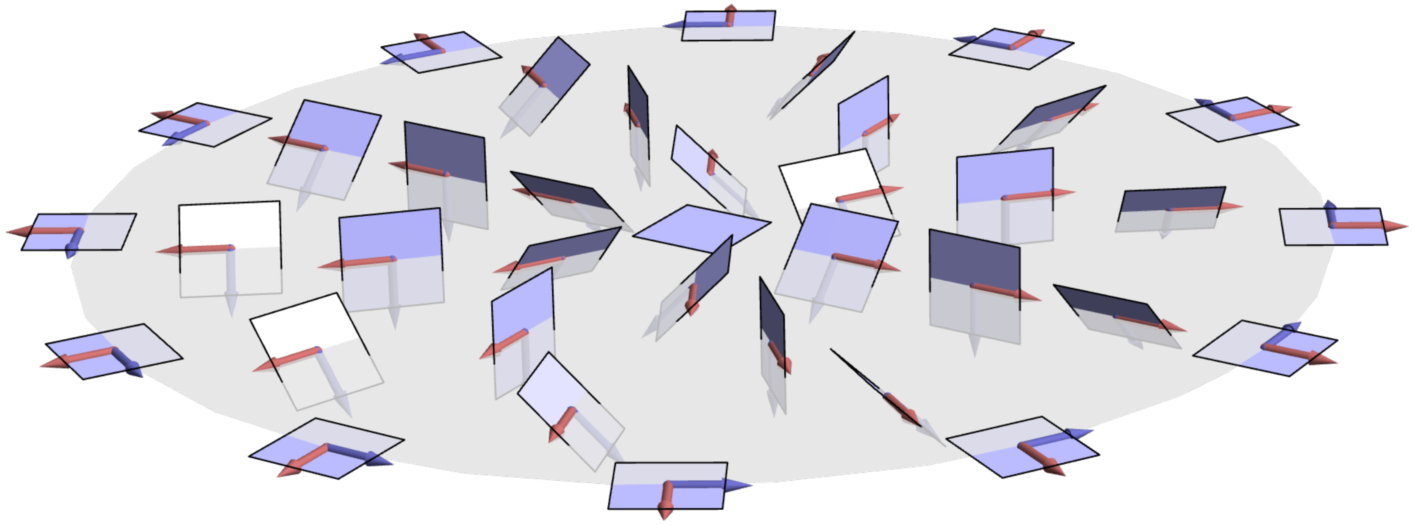
form an orthonormal basis for the planes perpendicular to $\mathbf{n}$. They are defined at all points except the centre of the disc, where they have a defect with winding number $+1$.

The connection is

$$\mathbf{e}_2 \cdot d\mathbf{e}_1 = \cos\frac{\pi r}{R}\,d\phi,$$

and the curvature is

$$\Omega = d\big(\mathbf{e}_2 \cdot d\mathbf{e}_1\big) = -\frac{\pi}{R}\sin\frac{\pi r}{R}\,dr \wedge d\phi = -\frac{1}{2}\,\mathbf{n}\cdot d\mathbf{n}\times d\mathbf{n}\,.$$



III. Skyrmions

The Gauss-Bonnet-Chern theorem reads

$$\int_{\partial D} \mathbf{e}_2 \cdot d\mathbf{e}_1 - \int_D \Omega = 2\pi \sum_i w_{p_i},$$

$$\Rightarrow \quad \int_0^{2\pi} \left(-1\right) d\phi - \left(-4\pi\right) = 2\pi\left(+1\right) . \qquad \checkmark$$

It can be considered undesirable to have boundary conditions that are not constant, so that a different choice of connection should be made to remove the winding and vanish everywhere on the disc boundary.

Exercise: Find a choice that does this and reexamine the Gauss-Bonnet-Chern theorem for your choice.

IV. Multi-skyrmion Textures and Energy Bound

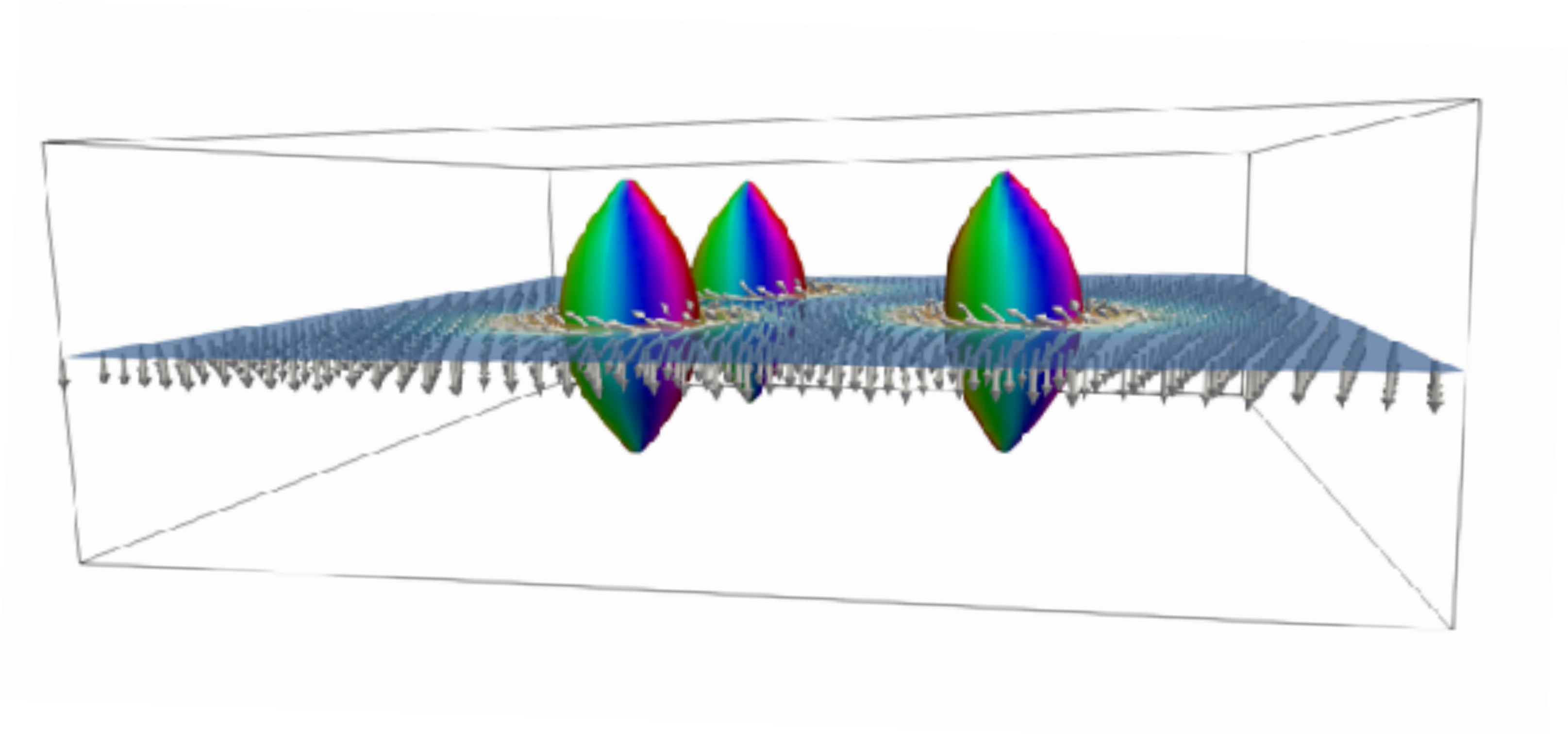
States with a **general number of skyrmions** can be constructed using the tools we have developed — a *planar texture with defects*; and *escape into the third dimension*.

We consider the director field

$$\mathbf{n} = \cos\theta\,\mathbf{e}_z + \sin\theta\left[\cos\eta\,\mathbf{e}_x + \sin\eta\,\mathbf{e}_y\right].$$

η has defects at isolated points p_i with winding numbers w_{p_i}, and vanishes (say) at large distances. We split the defects into two sets, $\mathscr{D}^+$ and $\mathscr{D}^-$, thinking of the former as the *locations of skyrmions* and the latter as whatever is needed to maintain the boundary conditions.

θ takes the value 0 at all defects in $\mathscr{D}^+$ (*escape up*) and the value π at all defects in $\mathscr{D}^-$ (*escape down*), and also at large distances; in between it interpolates smoothly.

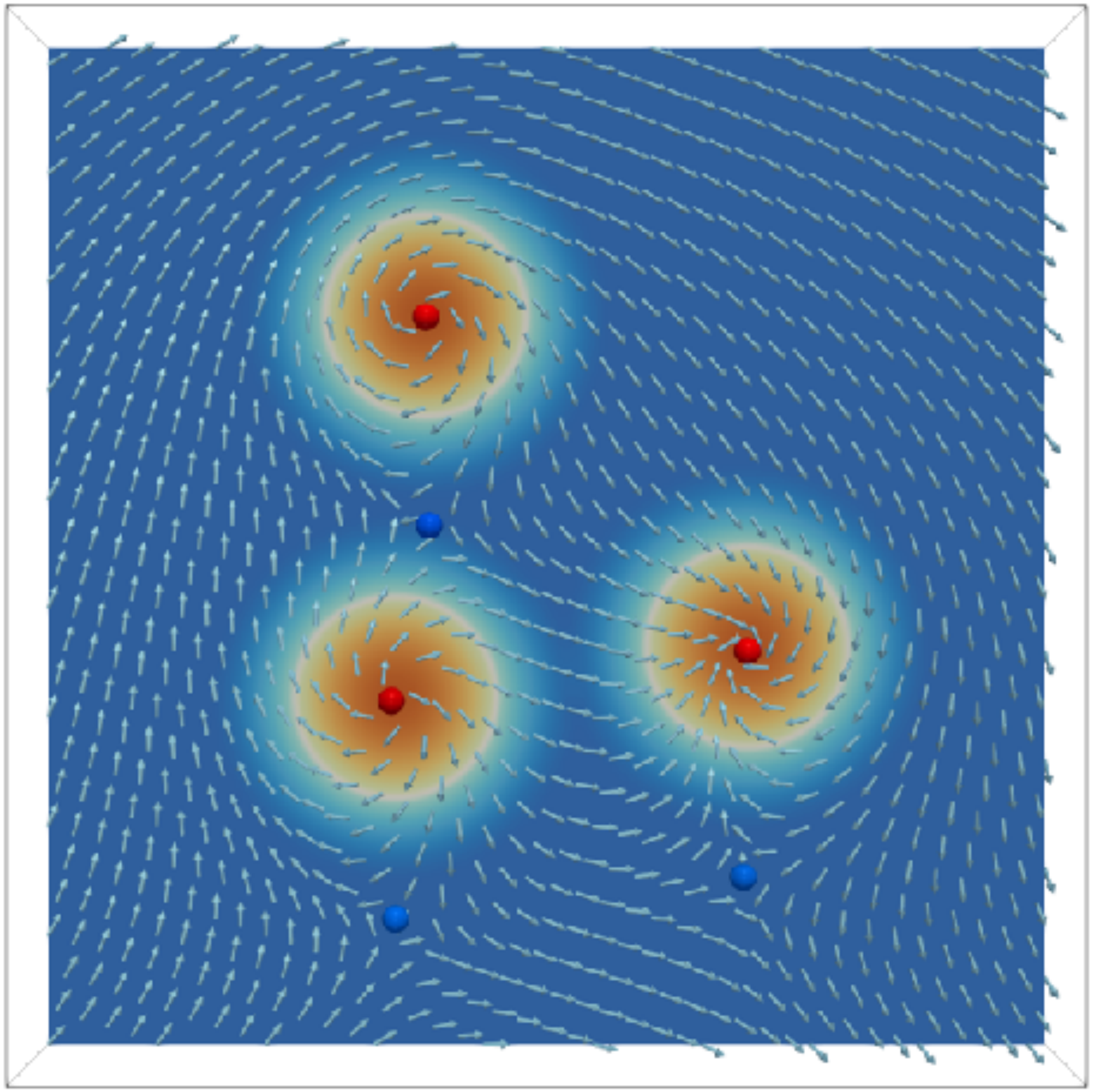


IV. Multi-skyrmion Textures and Energy Bound

$$\mathbf{n} = \cos\theta\,\mathbf{e}_z + \sin\theta\left[\cos\eta\,\mathbf{e}_x + \sin\eta\,\mathbf{e}_y\right].$$

η has defects at isolated points p_i with winding numbers w_{p_i}.

θ takes the value 0 at all defects in $\mathscr{D}^+$ (*escape up*) and the value π at all defects in $\mathscr{D}^-$ (*escape down*), and also at large distances.



IV. Multi-skyrmion Textures and Energy Bound

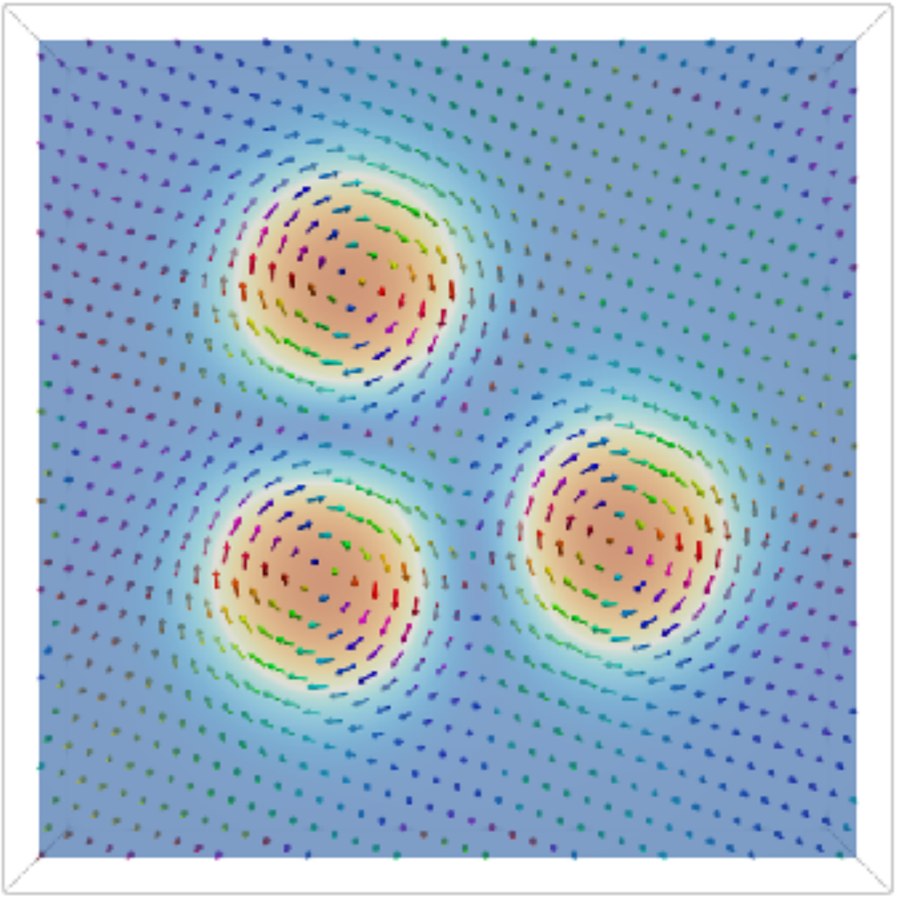
We compute the **skyrmion number** using the integral formula:

$$\int \mathbf{n}\cdot d\mathbf{n}\times d\mathbf{n} = 2\int \sin\theta\, d\theta\wedge d\eta = 2\int d\big(-\cos\theta\, d\eta\big) = 4\pi\Bigg(\sum_{p\in\mathscr{D}^+} w_p - \sum_{p\in\mathscr{D}^-} w_p\Bigg).$$

The constant boundary conditions on η imply the *sum of all the winding numbers is zero*. Hence

$$\int \mathbf{n}\cdot d\mathbf{n}\times d\mathbf{n} = 8\pi \sum_{p\in\mathscr{D}^+} w_p = 8\pi Q,$$

where Q is the total number of skyrmions.



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Belavin & Polyakov [JETP Lett. **22**, 245 (1975)] made the interesting observation that the one-elastic-constant free energy

$$F = \int \frac{K}{2} |\nabla \mathbf{n}|^2 \, dA = \frac{K}{2} \int |\nabla\theta|^2 + \sin^2\theta \, |\nabla\eta|^2 \, dA,$$

is **bounded from below** by the *skyrmion number*.

First, consider the skyrmion density and note

$$\int \mathbf{n} \cdot d\mathbf{n} \times d\mathbf{n} = 2 \int \sin\theta \, d\theta \wedge d\eta = 2 \int \sin\theta \left(\partial_x\theta \, \partial_y\eta - \partial_y\theta \, \partial_x\eta \right) dx \wedge dy \, .$$

Let J denote the **complex structure** in the xy-plane (*i.e.* the generator of rotations about the z-axis) and write this as

$$\int \mathbf{n} \cdot d\mathbf{n} \times d\mathbf{n} = -2 \int \sin\theta \, \nabla\theta \cdot J \nabla\eta \, dA \, .$$

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Next, we use $|\nabla\eta| = |J\nabla\eta|$ and write

$$F = \frac{K}{2}\int |\nabla\theta|^2 + \sin^2\theta\,|\nabla\eta|^2\,dA = \frac{K}{2}\int |\nabla\theta + \sin\theta\,J\nabla\eta|^2 - 2\sin\theta\,\nabla\theta\cdot J\nabla\eta\,dA,$$

$$= \frac{K}{2}\int |\nabla\theta + \sin\theta\,J\nabla\eta|^2\,dA + 4\pi K\,Q \geq 4\pi K\,Q\,.$$

This is an example of a **BPS bound** (after Bogomol'nyi and Prasad & Sommerfeld); the bound is attained for states that satisfy the **BPS equation**

$$\nabla\theta + \sin\theta\,J\nabla\eta = 0.$$

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$$\nabla\theta + \sin\theta\, J\, \nabla\eta = 0.$$

Belavin & Polyakov (1975) showed that this is equivalent to the Cauchy-Riemann equations for the complex function

$$\psi = \tan\frac{\theta}{2}\, \mathrm{e}^{i\eta} = \frac{\sin\theta\, \mathrm{e}^{i\eta}}{1+\cos\theta},$$

which coincides with stereographic projection of the director from the south pole.

Exercise: Verify these statements.

Thus, for a given number of skyrmions Q, the energy is **minimised** when ψ is a **meromorphic function** of $x + iy$.

Lecture 1: Textures in the Plane

Lecture 2: Escape from the Plane

Lecture 3: Hopfions and Chiral Topology

Lecture 4: Practicals — examples & discussion

I. Meyer's Escape into the Third Dimension

II. The Degree of Point Defects

III. Skyrmions

IV. Multi-skyrmion Textures and Energy Bound