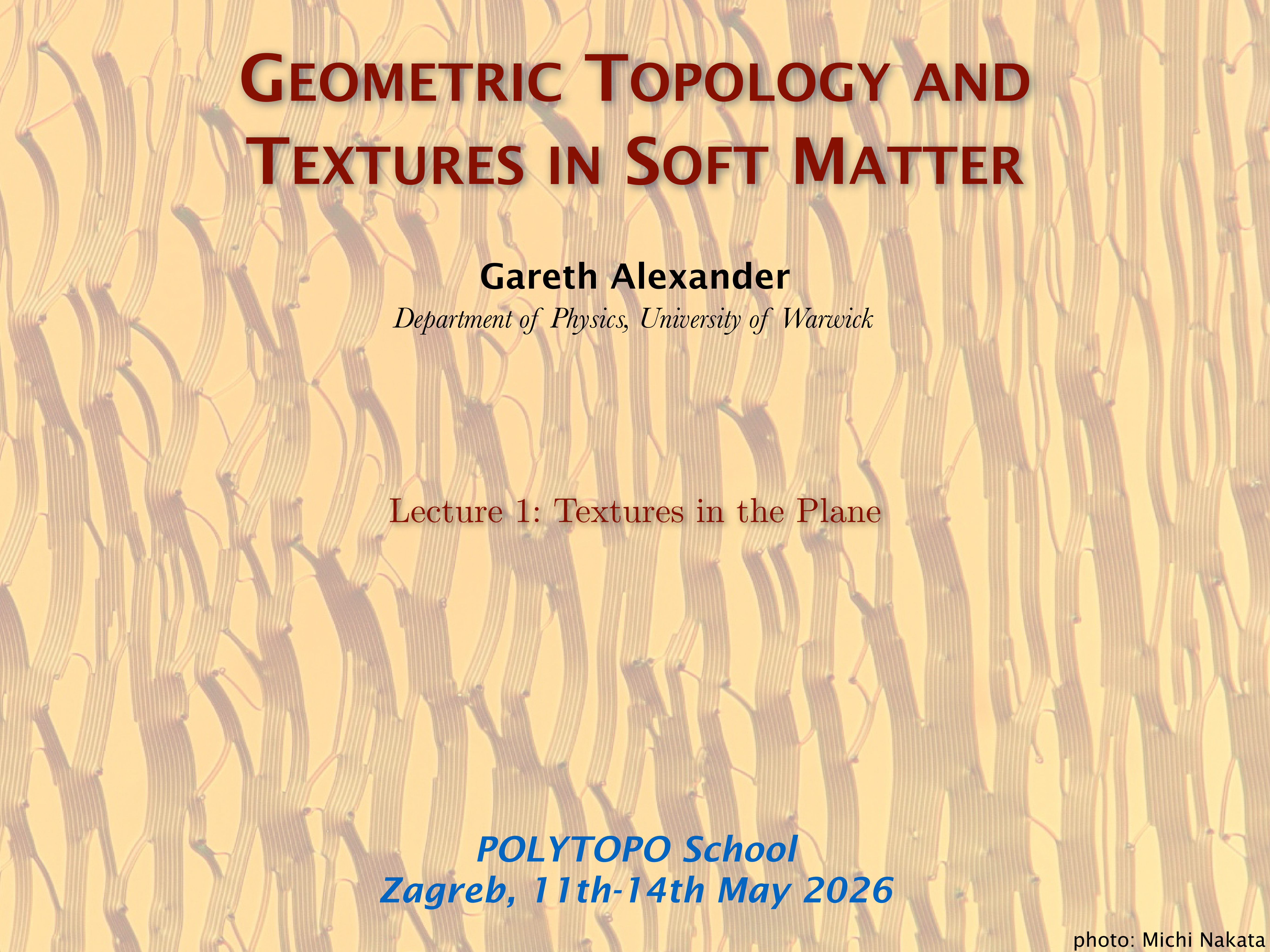


GEOMETRIC TOPOLOGY AND TEXTURES IN SOFT MATTER

Gareth Alexander
Department of Physics, University of Warwick

Lecture 1: Textures in the Plane

POLYTOPO School
Zagreb, 11th-14th May 2026

photo: Michi Nakata

Lecture 1: Textures in the Plane

Lecture 2: Escape from the Plane

Lecture 3: Hopfions and Chiral Topology

Lecture 4: Practicals — examples & discussion

I. Textures and their Defects

II. Connection and Winding Numbers

III. Schlieren Textures and Pontrjagin-Thom Construction

IV. On the exterior calculus

V. A Digression onto Curved Surfaces

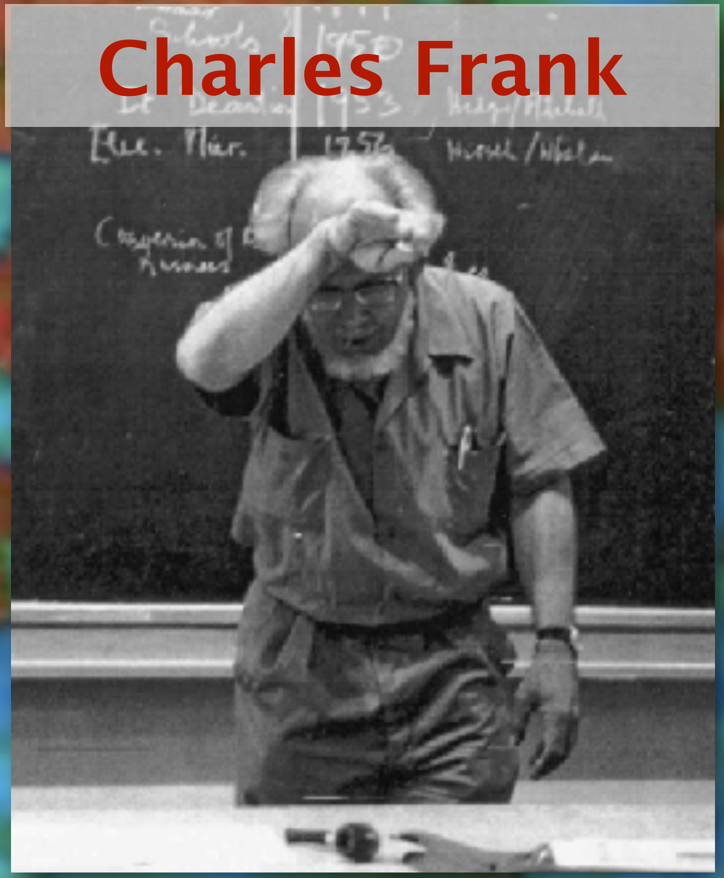
I. Textures and their Defects

Google's AI overview: *(emphasis is its own)*

Texture refers to the tactile quality of a surface—how it feels when touched—or the visual representation of that quality. It describes characteristics such as rough, smooth, soft, or hard, and is a key element in art, design, and food to add realism, functionality, or sensory experience.

On Topological Textures …

“They are totally useless, I think, except for one important intellectual use, that of providing tangible examples of topological oddities and so helping to bring topology into the public domain of science, from being the private preserve of a few abstract mathematicians and particle physicists”



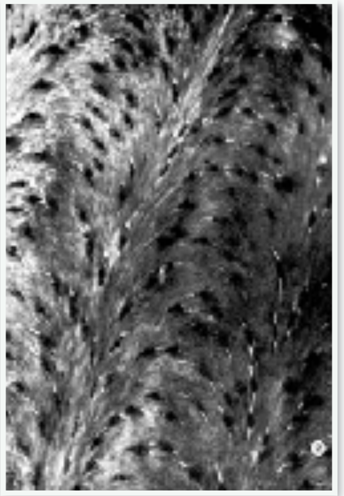
Charles Frank

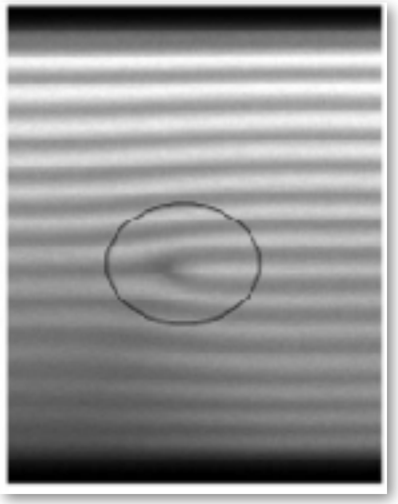
I. Textures and their Defects

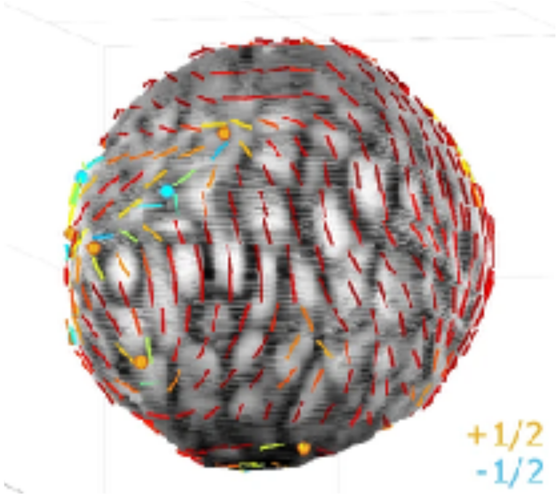
I will describe textures of **orientational order**, *e.g.* magnetic fields, nematic liquid crystals, *etc*. At each point of the material there is an **orientation**, or **direction** in space.

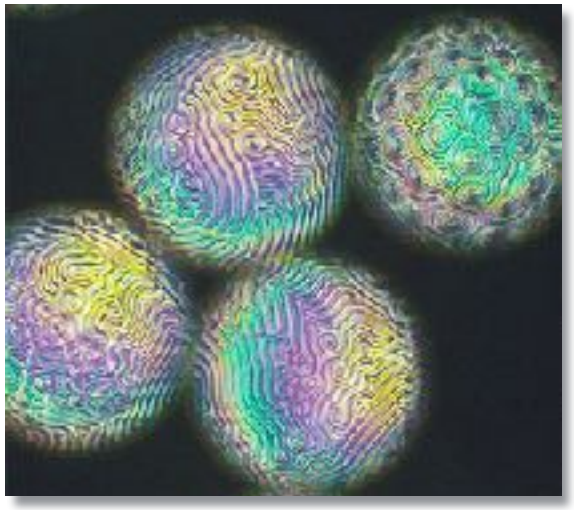
The **texture** is what this ‘looks like’ or ‘feels like’. It will be given by a **unit magnitude vector field**, $\mathbf{n}$, called the **director**. It has **defects** at a set of isolated points $\mathscr{D}$. Away from these it is **smooth**[†].

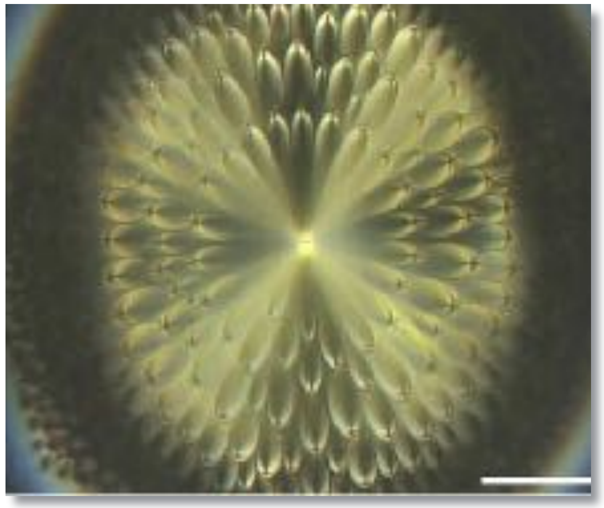
In the nematic case the orientation is **apolar**, $\mathbf{n} \sim -\mathbf{n}$. We will call this **nematic symmetry**. It has many interesting properties but for the most part we will assume a proper vector and treat nematic symmetry only in a few passing remarks.











[†] I will not address any questions of regularity and so just use ‘smooth’ here. The defects are points where there is a loss of continuity.

I. Textures and their Defects

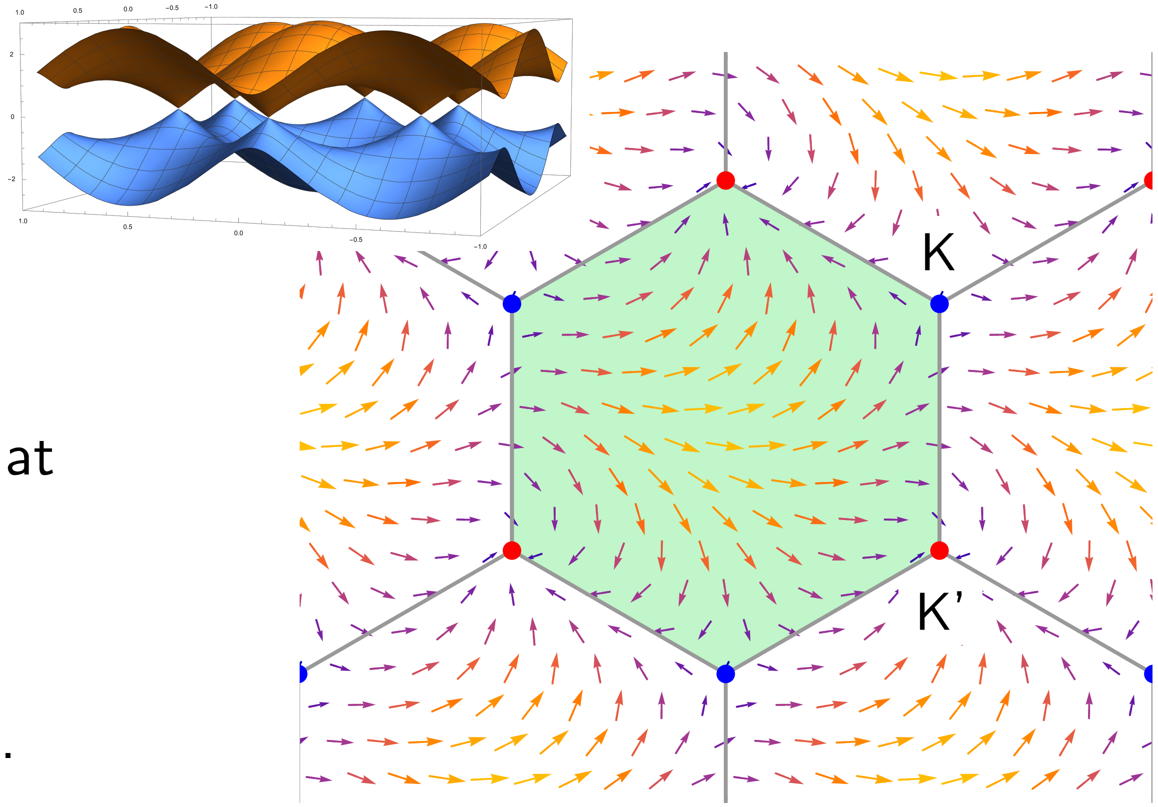
Other physical systems can also be described as textures of **orientational order**.

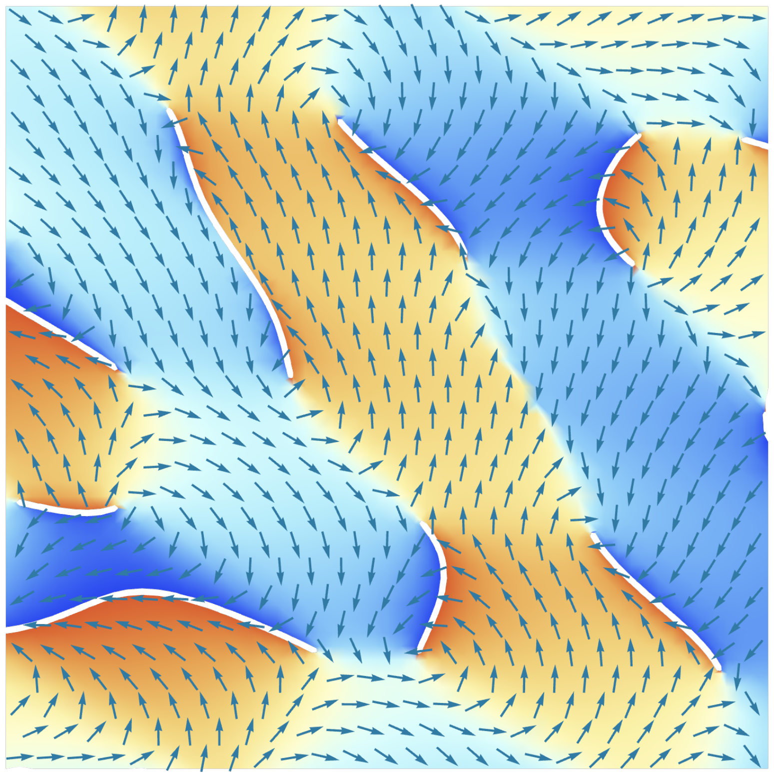
Band Hamiltonian

$$H = E_g\,\mathbf{n}\cdot\boldsymbol{\sigma} = E_g \begin{bmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{bmatrix},$$

corresponds to an energy gap E_g and orientation $\mathbf{n}$ at each point of the Brioullin Zone.

Band structure of graphene described by hopping element $t(\mathbf{k}) = 1 + \mathrm{e}^{i(\sqrt{3}k_x + k_y)/2} + \mathrm{e}^{ik_y} = E_g(n_x + i n_y)$.





Complex scalar field $\psi : \mathbb{R}^2 \to \mathbb{C}$ *(wavefunction, optics)*

$$\psi(x,y) = \psi_0(x,y)\,\mathrm{e}^{i\theta(x,y)}.$$

Phase θ defines an orientation and can be encoded in a unit vector $n_x + i n_y = \mathrm{e}^{i\theta}$.

E.g. a sum of plane waves with random wavevectors.

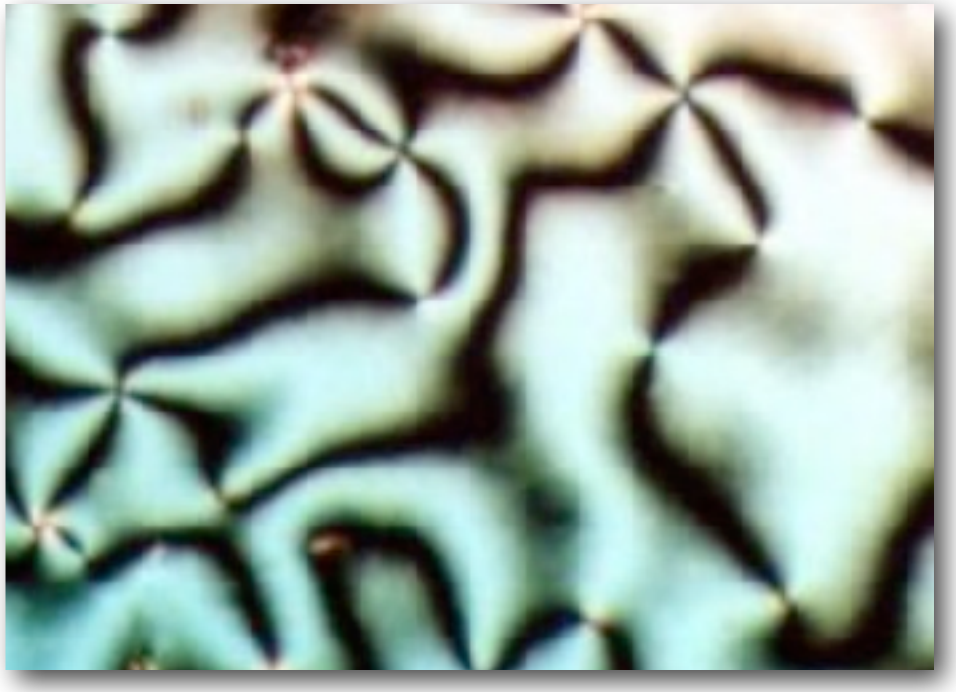
I. Textures and their Defects

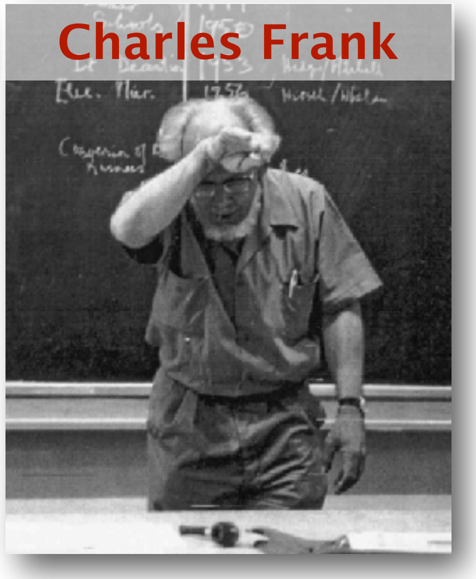
Textures have **defects** at a set of isolated points $\mathscr{D}$. These are points where there is a **loss of continuity**.

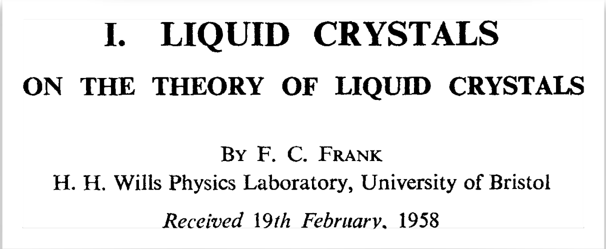
For a complex scalar, a minimal model is given by zeros of order k (in $x + iy$ or $x - iy$)

$$\psi \sim (x + iy)^k, \qquad \Rightarrow \qquad \theta \sim k \, \mathrm{Im} \ln(x + iy) = k \arctan \frac{y}{x} \, .$$

The same local models also describe defects in liquid crystals.







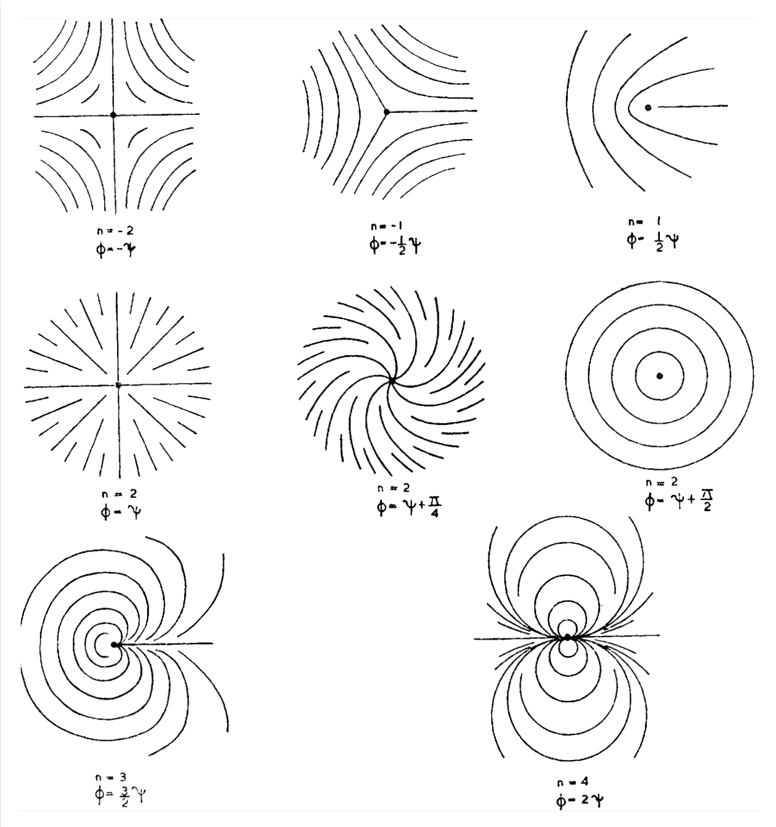
I. LIQUID CRYSTALS

ON THE THEORY OF LIQUID CRYSTALS

By F. C. Frank

H. H. Wills Physics Laboratory, University of Bristol

Received 19th February, 1958

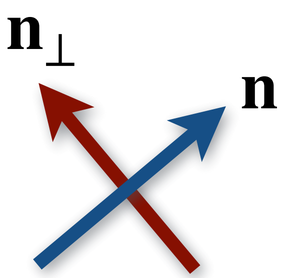


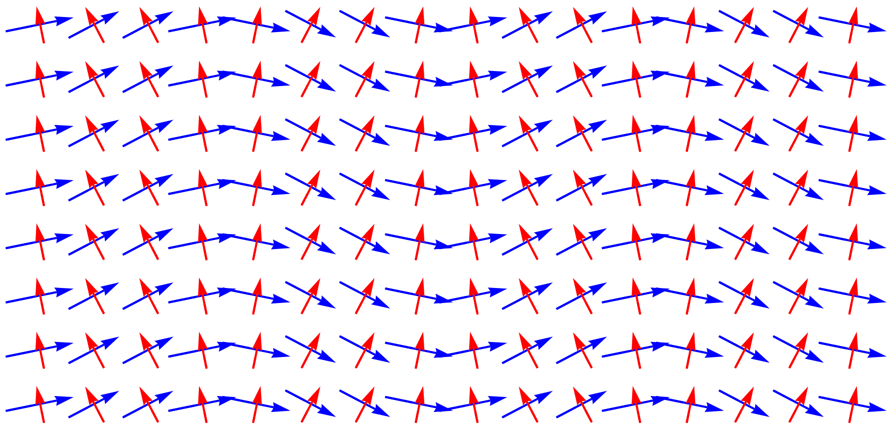
II. Connections and Winding Numbers

The **texture** is given by a **unit magnitude vector field**, $\mathbf{n}$, called the **director**.

The director is a unit vector, $\mathbf{n} \cdot \mathbf{n} = 1$, so $d\big(\mathbf{n} \cdot \mathbf{n}\big) = 2\,\mathbf{n} \cdot d\mathbf{n} = 0$.

Hence $d\mathbf{n}$ points in the direction $\mathbf{n}_\perp$ perpendicular to $\mathbf{n}$.



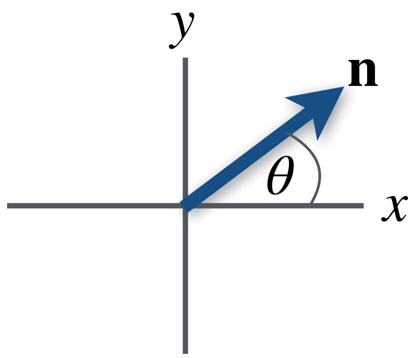


We may write $d\mathbf{n} = \omega \otimes \mathbf{n}_\perp$ but will more typically use the differential form $\mathbf{n}_\perp \cdot d\mathbf{n}$. It is called the **connection**.

Integrals of the connection around loops in the texture measure winding numbers.

II. Connections and Winding Numbers

Relative to the Cartesian basis we can write $\mathbf{n} = \cos\theta\, \mathbf{e}_x + \sin\theta\, \mathbf{e}_y$, where $\theta = \theta(x, y)$ is the angle the director makes with the x-axis.



The connection is the differential form

$$\mathbf{n}_\perp \cdot d\mathbf{n} = d\theta = \frac{\partial\theta}{\partial x}\, dx + \frac{\partial\theta}{\partial y}\, dy,$$

and measures the rate the director rotates relative to the Cartesian basis.

Its integral around any closed curve C is the total rotation of the director; continuity implies this total rotation must be an integer multiple of 2π

$$\int_C \mathbf{n}_\perp \cdot d\mathbf{n} = \int_C d\theta = 2\pi\, w(C)\,.$$

We call $w(C)$ the **winding number** of the curve C.

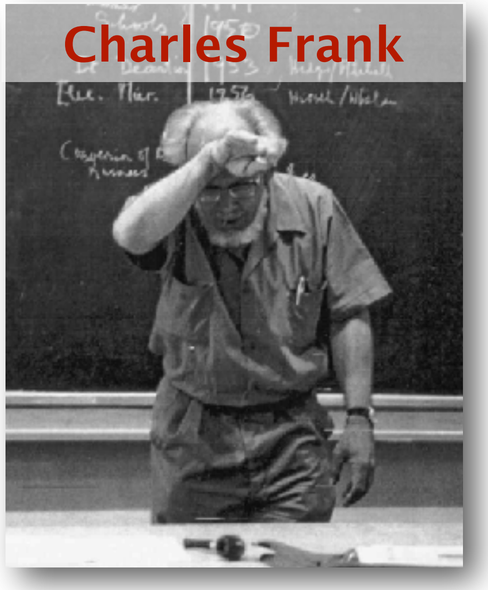
II. Connections and Winding Numbers

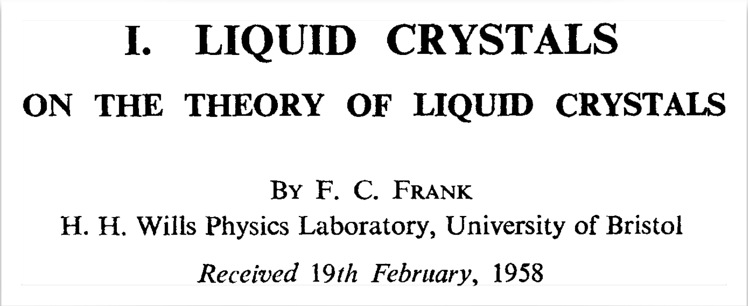
Its integral around any closed curve C is the total rotation of the director; continuity implies this total rotation must be an integer multiple of 2π

$$\int_C \mathbf{n}_\perp \cdot d\mathbf{n} = \int_C d\theta = 2\pi\, w(C)\,.$$

We call $w(C)$ the **winding number** of the curve C.

In liquid crystals the **nematic symmetry** $\mathbf{n} \sim -\mathbf{n}$ allows for the total rotation to be a half-integer multiple of 2π and in this case the winding number is any **half-integer**.





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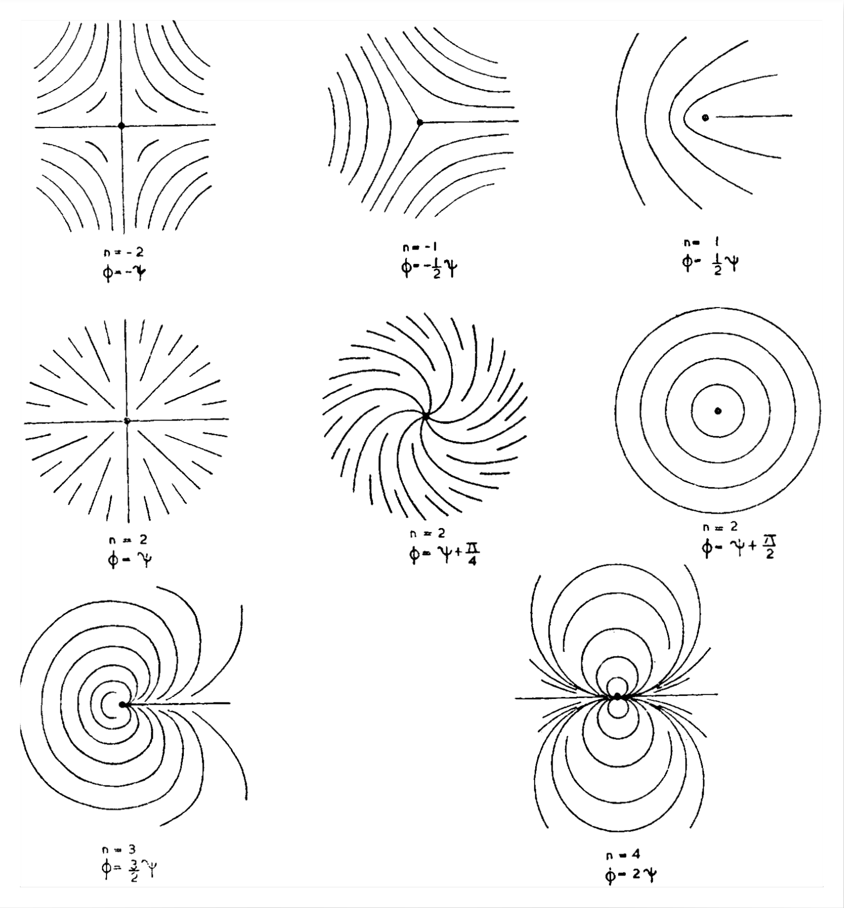
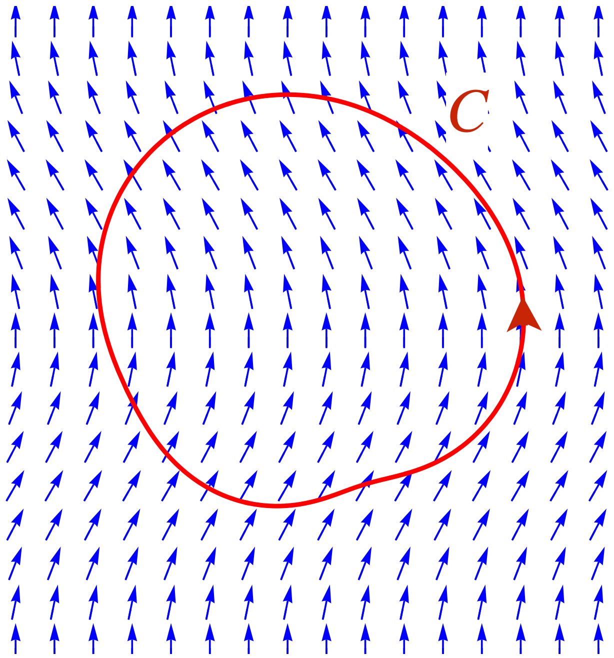


II. Connections and Winding Numbers

Suppose C encloses a region R with **no defects**. Then the winding number is **zero**.



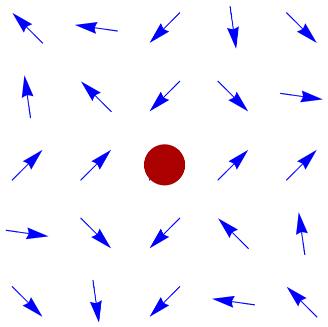
A way of showing this is to use Green's theorem in the plane

$$\int_{\partial R} P\,dx + Q\,dy = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy\,.$$

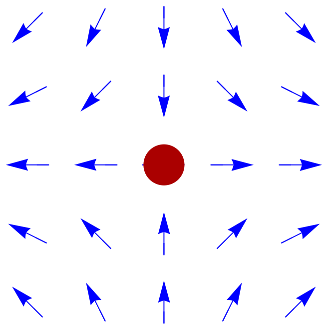
In our case $P = \partial\theta/\partial x$, $Q = \partial\theta/\partial y$ and the integrand on the RHS vanishes.

Winding numbers are only non-zero when there are defects.

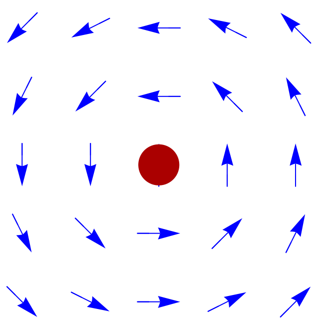
II. Connections and Winding Numbers



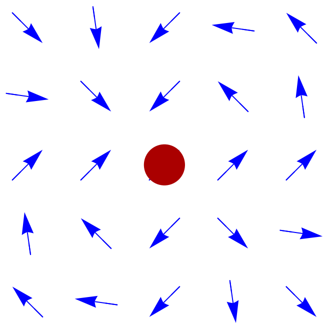
$k = -2$



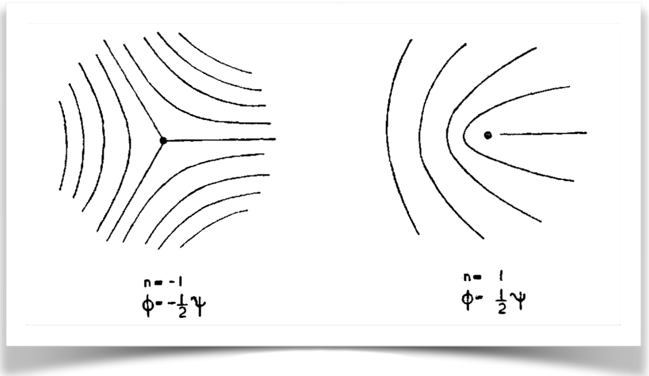
$k = -1$



$k = 1$



$k = 2$



Consider the local models for a single point defect at the origin

$$\theta = k \arctan \frac{y}{x} + \theta_0 .$$

Away from the origin the connection is

$$\mathbf{n}_\perp \cdot d\mathbf{n} = d\theta = k \frac{-y\,dx + x\,dy}{x^2 + y^2},$$

and integrating around a circle of radius r ($x = r\cos\phi\,, y = r\sin\phi\,, \phi \in [0{,}2\pi)$), the winding number is

$$w(C) = \frac{1}{2\pi}\int_C k\frac{-y\,dx + x\,dy}{x^2 + y^2} = \frac{k}{2\pi}\int_0^{2\pi} d\phi = k\,.$$

This is independent of the radius and suggests the winding number can be viewed as a **property of the defect**.

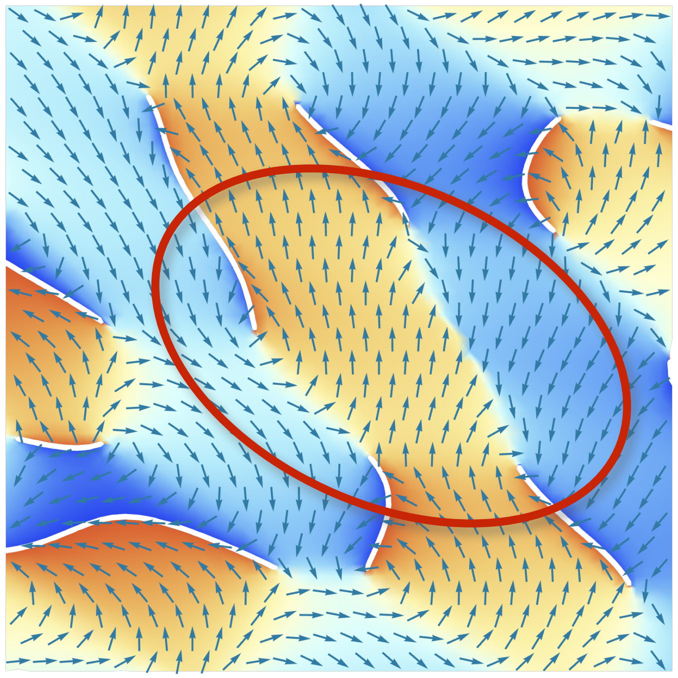
II. Connections and Winding Numbers

$$w(C) = \frac{1}{2\pi}\int_C k\,\frac{-y\,dx + x\,dy}{x^2+y^2} = \frac{k}{2\pi}\int_0^{2\pi} d\phi = k\,.$$

This is independent of the radius and suggests the winding number can be viewed as a **property of the defect**.

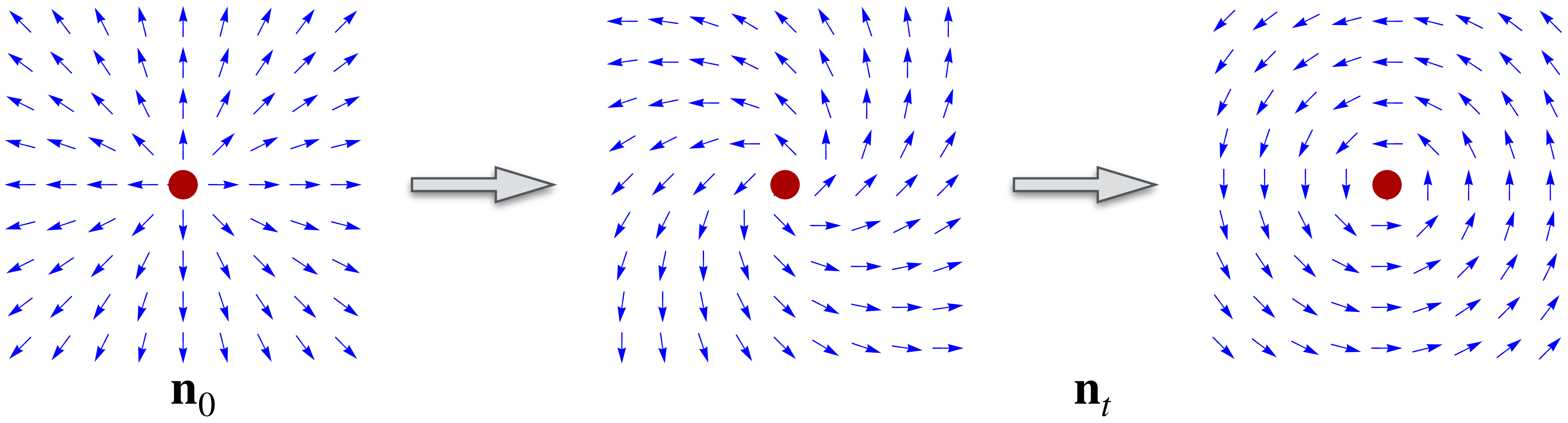
Exercise: For a curve C enclosing isolated defects at points p_i with winding numbers w_{p_i}

$$w(C) = \sum_i w_{p_i}\,.$$



II. Connections and Winding Numbers

Winding numbers are **preserved** under **continuous evolution** of the texture.



Let $\mathbf{n}_t$ be the texture at time t and $\mathbf{n}_0$ the initial texture. Since $\mathbf{n}_0, \mathbf{n}_{0\perp}$ form a basis we can write (away from any defects)

$$\mathbf{n}_t = \cos\alpha\,\mathbf{n}_0 + \sin\alpha\,\mathbf{n}_{0\perp},$$

where α is a continuous function (of position and time) that vanishes identically at $t = 0$.

The connection is $\mathbf{n}_{t\perp} \cdot d\mathbf{n}_t = \mathbf{n}_{0\perp} \cdot d\mathbf{n}_0 + d\alpha$, and the winding number is

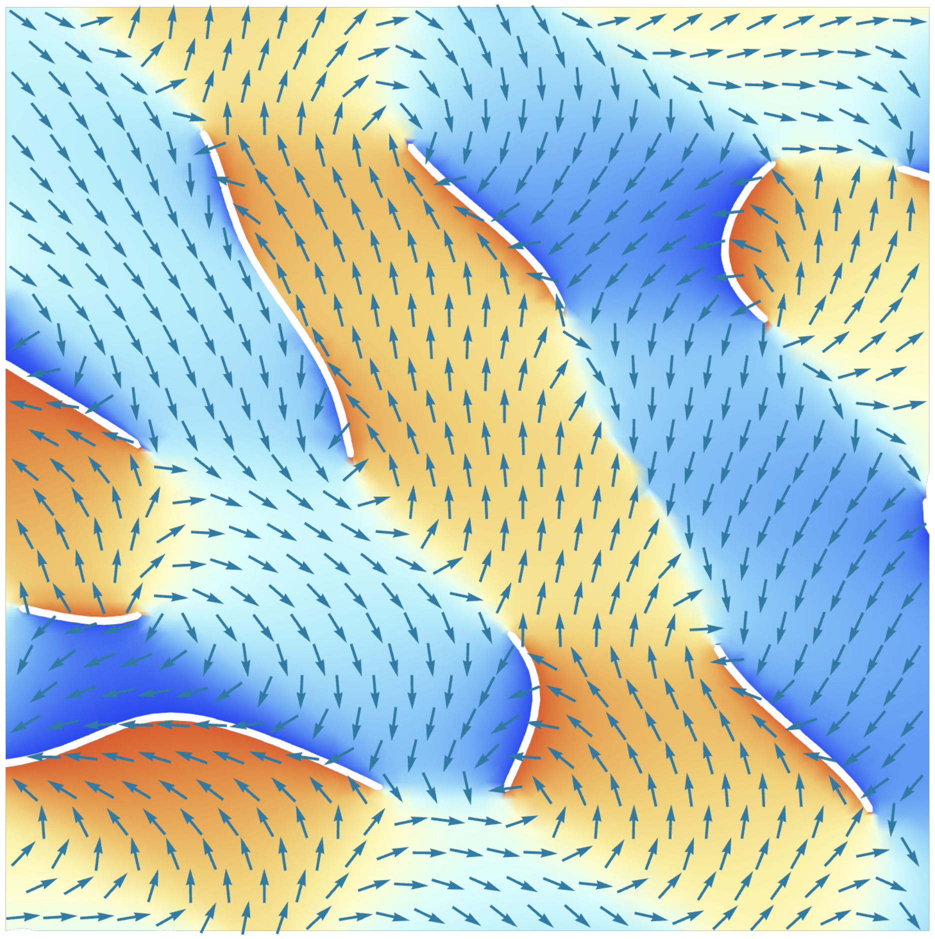
$$w_t(C) = \frac{1}{2\pi}\int_C \mathbf{n}_{t\perp} \cdot d\mathbf{n}_t = w_0(C) + \frac{1}{2\pi}\int_C d\alpha\,.$$

Since α is a function $\displaystyle\int_C d\alpha$ vanishes by the fundamental theorem of calculus.

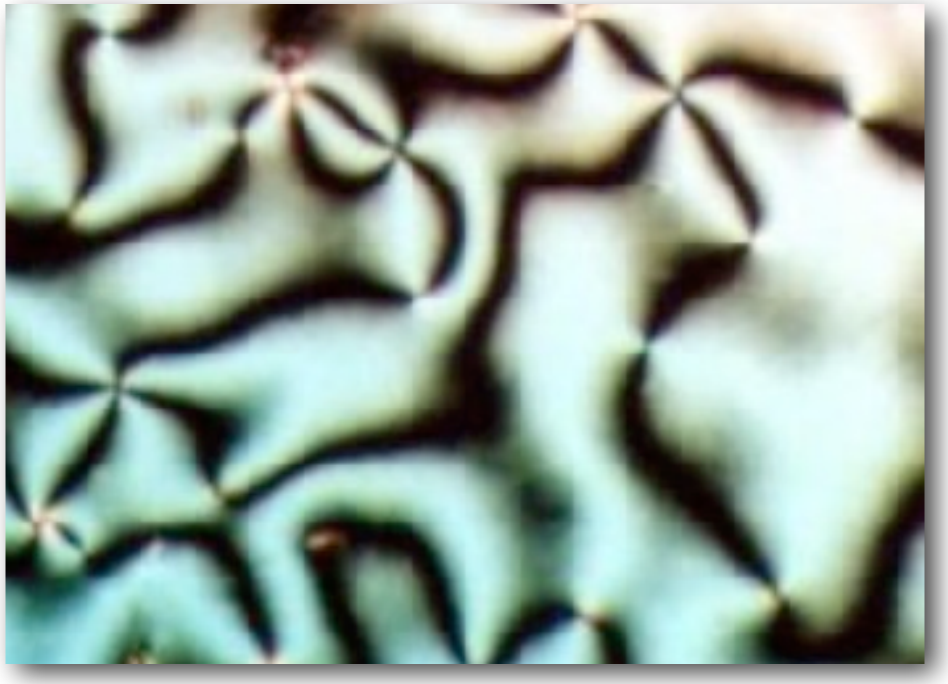
II. Connections and Winding Numbers

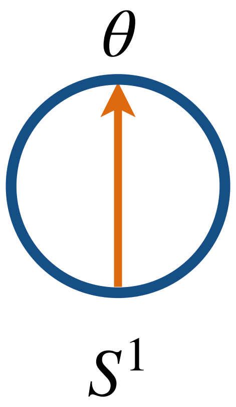
Winding numbers are preserved integer (or half-integer) labels for defects in planar textures.

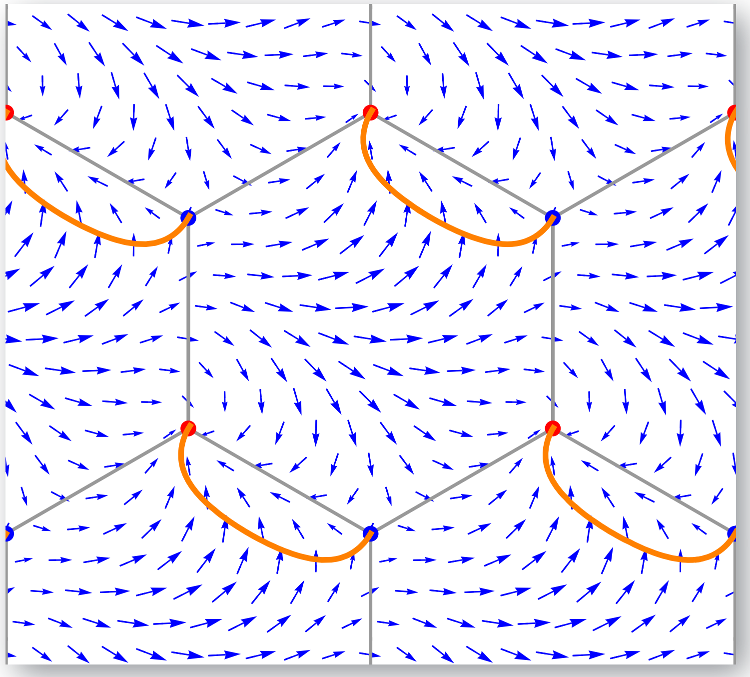
For planar textures the connection $\mathbf{n}_{\perp} \cdot d\mathbf{n}$ represents a cohomology class in $H^1(\mathbb{R}^2 \backslash \mathscr{D})$ that captures the topology of the texture.



III. Schlieren Textures and Pontrjagin-Thom Construction







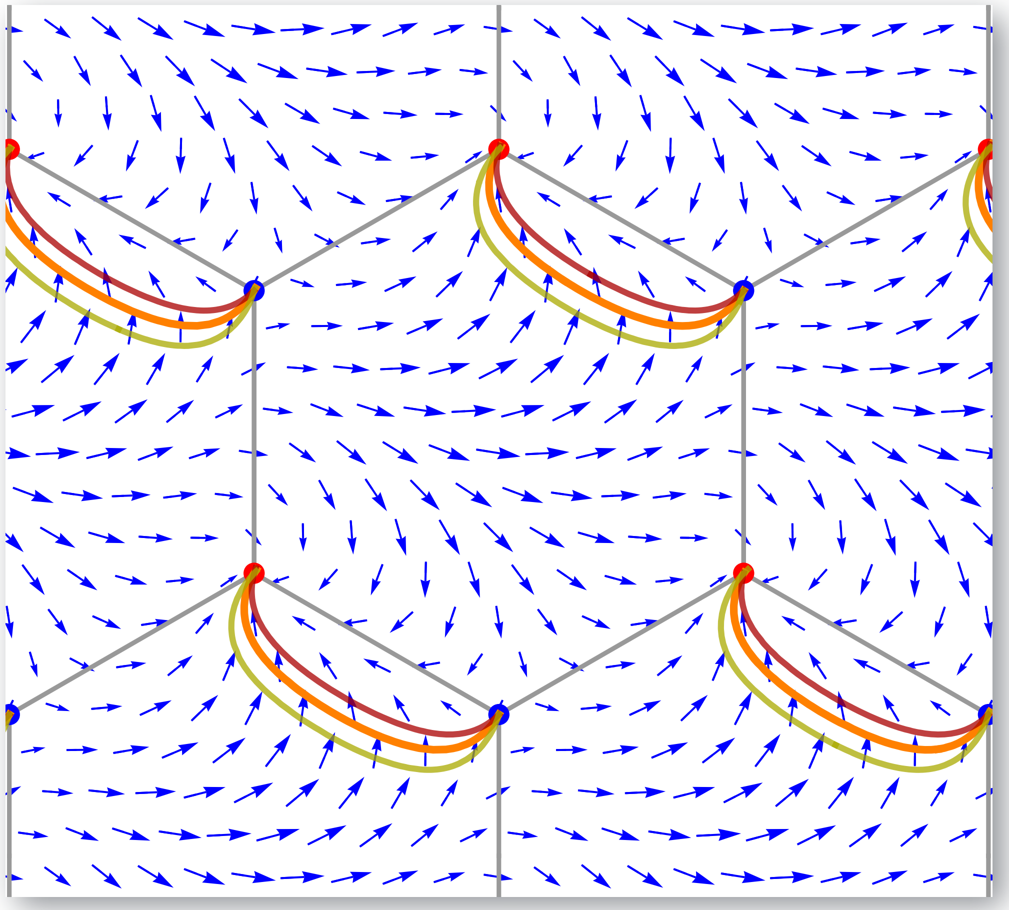
The texture is an assignment of an orientation to each point of the material away from the defects

$$\begin{aligned} \mathbf{n} : \mathbb{R}^2 \backslash \mathscr{D} &\to S^1 \,. \\ \mathbf{x} &\mapsto \theta \end{aligned}$$

Fix a point of S^1, *i.e.* an orientation θ. The **inverse image** $\mathbf{n}^{-1}(\theta)$ is the set of points in the texture where the director $\mathbf{n}$ has that orientation.

For almost any choice of θ the inverse image is 'nice' — a submanifold with end points only on the defects; this is called a **regular value**.

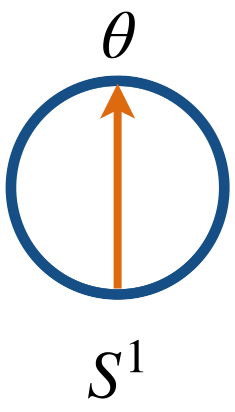
III. Schlieren Textures and Pontrjagin-Thom Construction

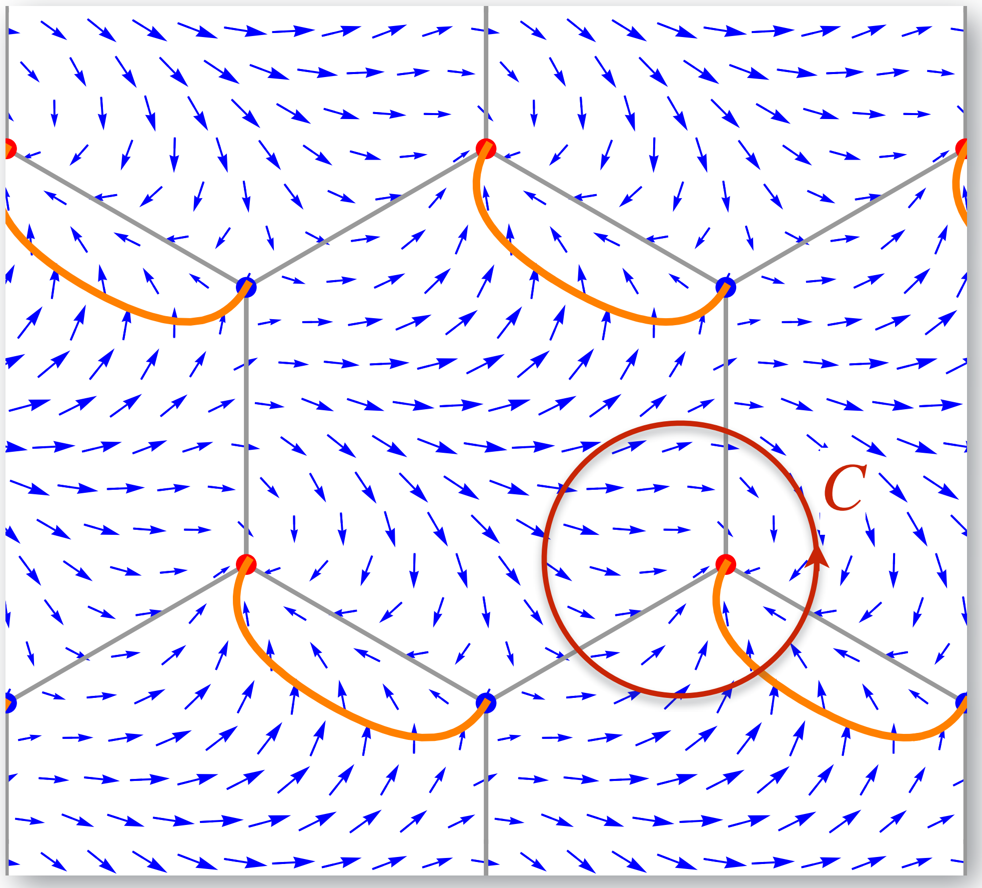


The **inverse image** $\mathbf{n}^{-1}(\theta)$ is the set of points in the texture where the director $\mathbf{n}$ has that orientation.

By continuity, nearby orientations have nearby inverse images. The inverse image is **cooriented** by the direction it moves in if θ is increased by a small amount.

III. Schlieren Textures and Pontrjagin-Thom Construction





A closed curve C will intersect the inverse image $\mathbf{n}^{-1}(\theta)$ in a discrete set of points. Each intersection can be assigned a sign ($\pm$) according to whether the orientation of C and the inverse image are the same, or opposite.

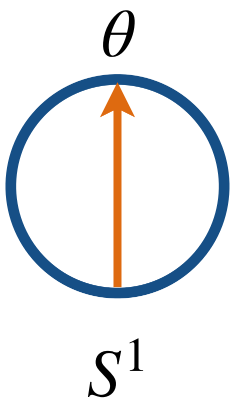
The **intersection number** $\mathrm{Int}\big(C, \mathbf{n}^{-1}(\theta)\big)$ is the sum of all these signs.

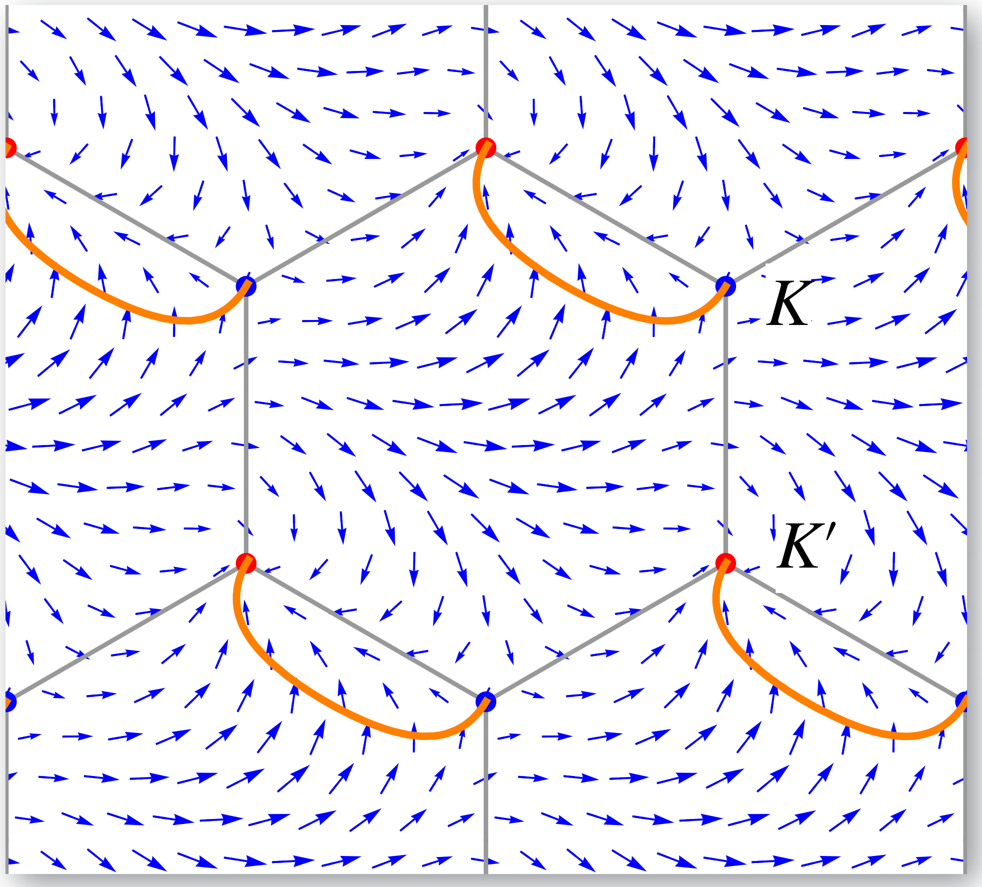
Part of the Pontrjagin-Thom theorem is that it equals the winding number

$$w(C) = \mathrm{Int}\big(C, \mathbf{n}^{-1}(\theta)\big) .$$

The information contained in the inverse image is enough to reconstruct the texture up to homotopy.

III. Schlieren Textures and Pontrjagin-Thom Construction





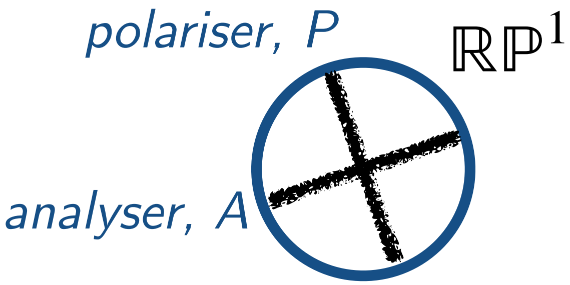
Exercise: Determine the winding numbers of the K and K' points using the Pontrjagin-Thom construction.

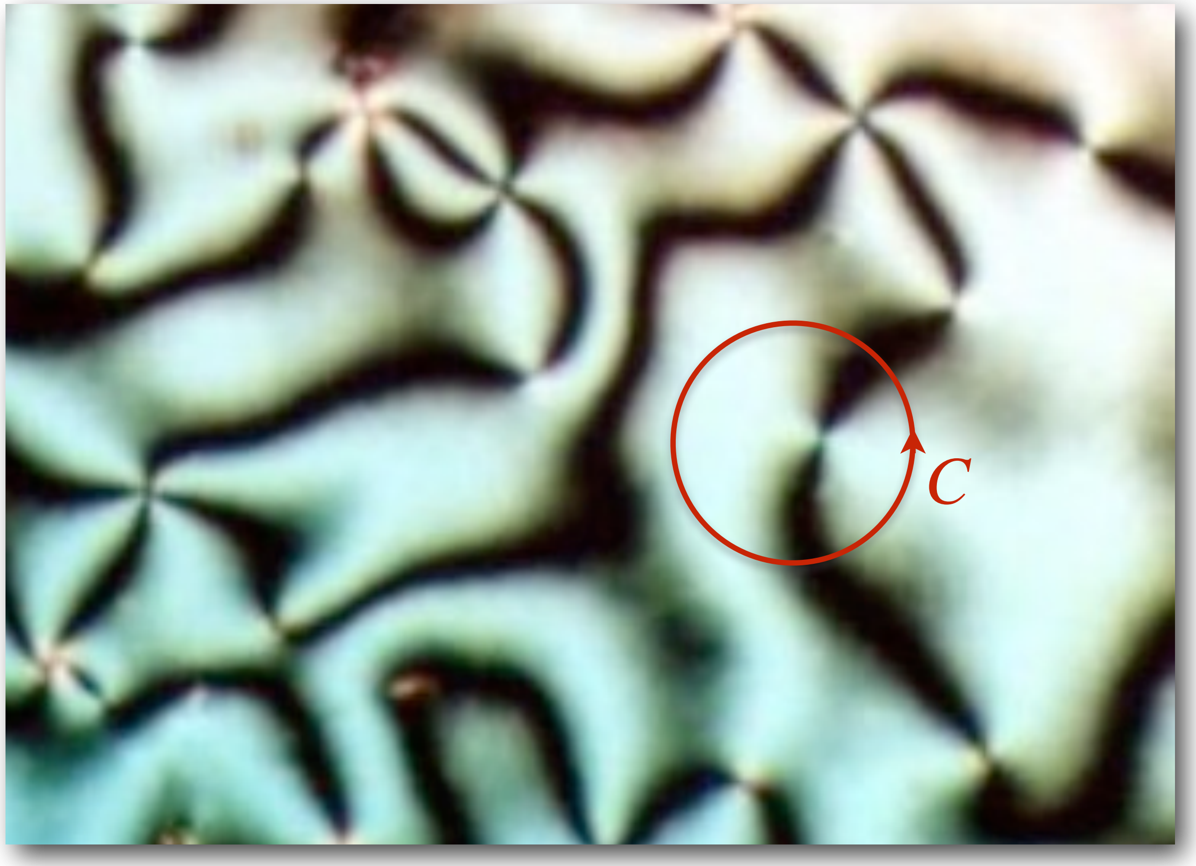
III. Schlieren Textures and Pontrjagin-Thom Construction

For nematic symmetry $\mathbf{n} \sim -\mathbf{n}$ and a single nematic orientation (line element) corresponds to the pair of antipodal points θ and $\theta + \pi$ on S^1[†]. This leads to a factor of $\frac{1}{2}$ in the relation between the intersection number and winding number

$$w(C) = \frac{1}{2}\,\mathrm{Int}\big(C, \mathbf{n}^{-1}(\theta)\big)\,.$$

But otherwise, everything remains the same.





[†] The space of line elements is the projective line $\mathbb{RP}^1$, which is the equivalence classes of antipodal points on S^1.

IV. On the Exterior Calculus

We provide a minimal introduction to the **exterior calculus**, presenting key ideas using Green's theorem in the plane

$$\int_{\partial R} P\,dx + Q\,dy = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy\,.$$

If the differential form $P\,dx + Q\,dy = df$, for some function f we say it is **exact**. We write

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy,$$

and think of the LHS as the **exterior derivative** d acting on f. For any exact form the integrand on the RHS of Green's theorem vanishes identically.

IV. On the Exterior Calculus

$$\int_{\partial R} P\,dx + Q\,dy = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy\,.$$

The idea is to view the RHS of Green's theorem as the action of the same operator, which we will write

$$d\big(P\,dx + Q\,dy\big)\,.$$

When the differential form is exact this should vanish identically

$$d\big(df\big) = 0, \qquad \Rightarrow \quad d^2 = 0.$$

This is the **key structural property**.

IV. On the Exterior Calculus

$$\int_{\partial R} P\,dx + Q\,dy = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy\,.$$

In general, we use the product rule (and $d^2 = 0$) to get

$$d\big(P\,dx + Q\,dy\big) = \left(\frac{\partial P}{\partial x}\,dx + \frac{\partial P}{\partial y}\,dy\right) \wedge dx + \left(\frac{\partial Q}{\partial x}\,dx + \frac{\partial Q}{\partial y}\,dy\right) \wedge dy,$$

where we have introduced a symbol $\wedge$ (pronounced '**wedge**') to represent the operation. Green's theorem is recovered if we define the wedge product to be **antisymmetric**

$$dx \wedge dy = -\,dy \wedge dx, \qquad dx \wedge dx = 0, \quad \text{etc.}$$

and take

$$\int_R f\,dx \wedge dy \quad \text{to be the same as} \quad \int_R f\,dx\,dy\,.$$

IV. On the Exterior Calculus

It is traditional to compactify the notation by writing $\omega = P\,dx + Q\,dy$ and to call

$$\int_{\partial R} \omega = \int_R d\omega$$

(the generalised) **Stokes' theorem**.

IV. On the Exterior Calculus

We describe also the notion of **de Rham cohomology**.

We have seen already the concept of an **exact form**, $\omega = df$. A counterpart is the concept of a **closed form**, one for which $d\omega = 0$.

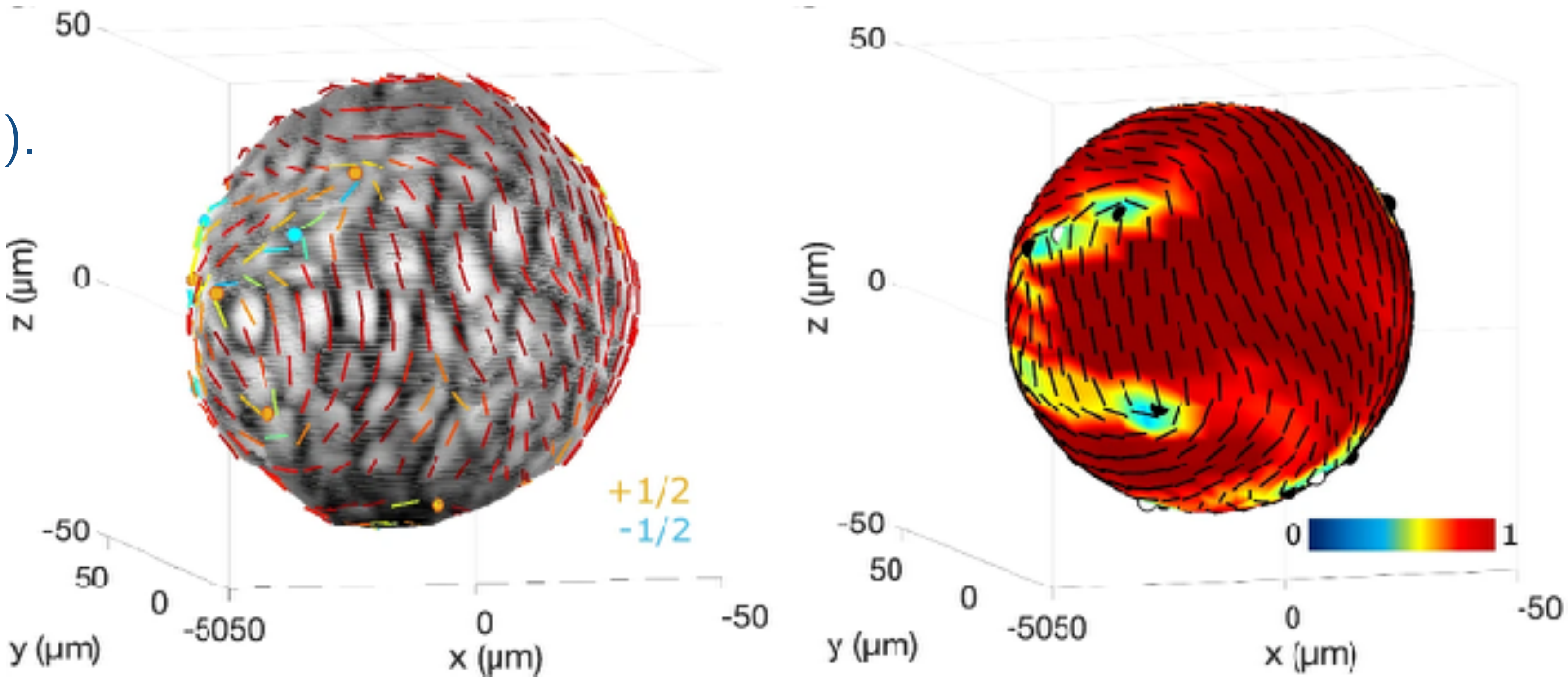
Forms that are exact are always closed. However, the converse is not always true.

Forms (in any dimension) that are **closed but not exact** constitute the **de Rham cohomology** (in that dimension). In the present discussion the forms are in dimension 1 and the de Rham cohomology is denoted H^1.

Exercise: The differential form $\omega = \dfrac{x\,dy - y\,dx}{x^2+y^2}$ is an element of cohomology of the plane minus the origin $\mathbb{R}^2 \setminus \{0\}$. Verify that it is closed. Verify that it is not exact.

V. A Digression onto Curved Surfaces

J. Eckert *et al.*
Nat. Comm. **16**, 7596 (2025).

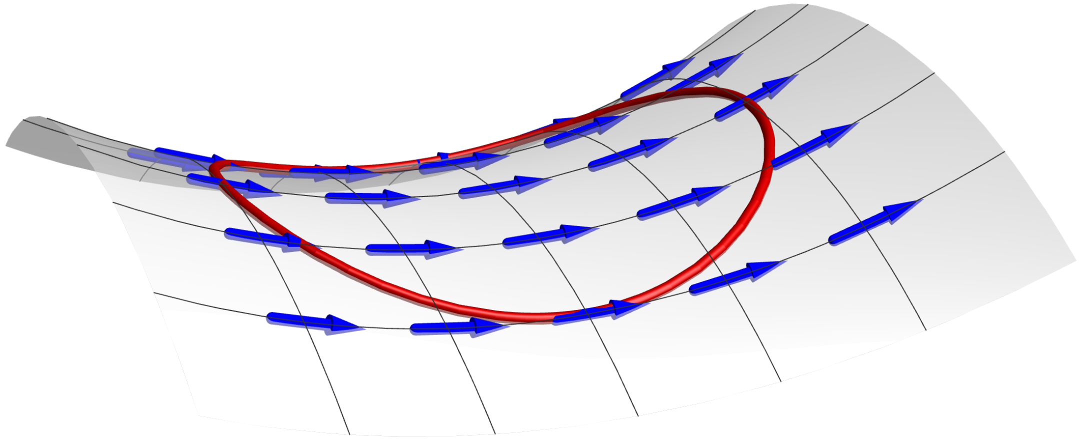


As in the plane we look at the connection $\mathbf{n}_\perp \cdot d\mathbf{n}$ and consider its integral around a closed curve C. For a region without any defects we have Stokes' theorem

$$\int_{\partial R} \mathbf{n}_\perp \cdot d\mathbf{n} = \int_R d\big(\mathbf{n}_\perp \cdot d\mathbf{n}\big)\,.$$

On a curved surface the integral on the RHS is no longer zero. This is tied to the curvature and for this reason we call the 2-form $\Omega = d(\mathbf{n}_\perp \cdot d\mathbf{n})$ the **curvature of the connection**.

V. A Digression onto Curved Surfaces



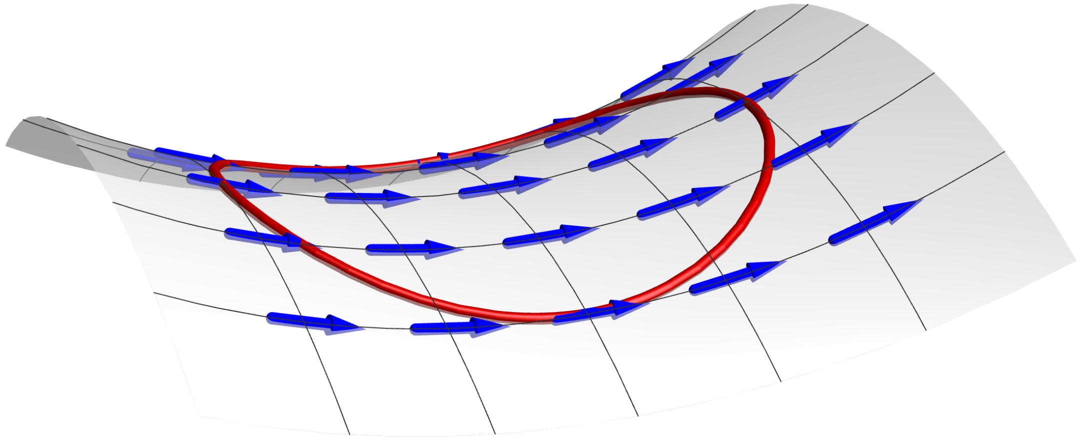
We compute it using locally adapted coordinates[†]. The surface is given by $(x, y, h(x, y))$ with height function (to quadratic order)

$$h(x, y) = \frac{1}{2}\left(k_1 x^2 + k_2 y^2\right) .$$

k_1, k_2 are the **principal curvatures** and x, y are the **principal curvature directions**.

[†] These are Riemann normal coordinates; in the theory of surfaces the representation by a height function is called Monge gauge.

V. A Digression onto Curved Surfaces



An orthonormal basis for the tangent space is (to linear order)

$$\mathbf{e}_1 = \mathbf{e}_x + k_1 x\,\mathbf{e}_z, \qquad\qquad \mathbf{e}_2 = \mathbf{e}_y + k_2 y\,\mathbf{e}_z\,.$$

We write the director as $\mathbf{n} = \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2$, and find

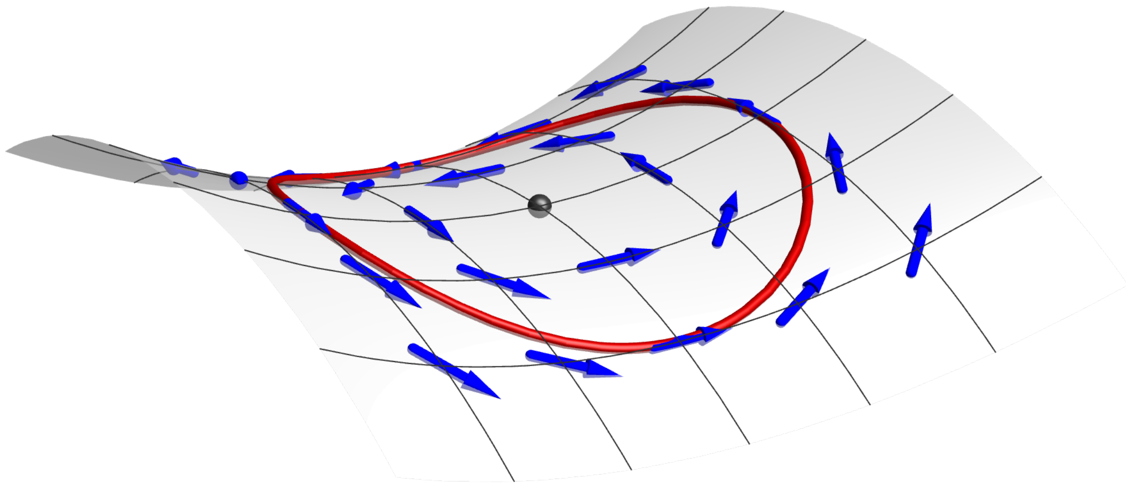
$$\mathbf{n}_\perp \cdot d\mathbf{n} = d\theta + k_1 k_2 y\,dx, \qquad \Rightarrow \qquad d\big(\mathbf{n}_\perp \cdot d\mathbf{n}\big) = k_1 k_2\,dy \wedge dx = -\,k_1 k_2\,dx \wedge dy\,.$$

The product $k_1 k_2 = K_{\mathsf{G}}$ is the **Gaussian curvature**.

Thus we find

$$\int_{\partial R} \mathbf{n}_\perp \cdot d\mathbf{n} = \int_R d\big(\mathbf{n}_\perp \cdot d\mathbf{n}\big) = -\int_R K_{\mathsf{G}}\,dA\,.$$

V. A Digression onto Curved Surfaces



Suppose there is an isolated defect at a point p and D is a small disc centred on it.

$\{\mathbf{e}_1, \mathbf{e}_2\}$ form a basis throughout D. On the **boundary** we can write $\mathbf{n} = \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2$. We then compute

$$\int_{\partial D} \mathbf{n}_\perp \cdot d\mathbf{n} = \int_{\partial D} \big(d\theta + \mathbf{e}_2 \cdot d\mathbf{e}_1\big) = \int_{\partial D} d\theta - \int_D K_{\mathsf{G}}\, dA\,.$$

The integral $\displaystyle\int_{\partial D} d\theta$ measures the winding of $\mathbf{n}$ relative to $\mathbf{e}_1$ and is equal to $2\pi\, w_p$, where w_p is the (integer) winding number at p.

V. A Digression onto Curved Surfaces

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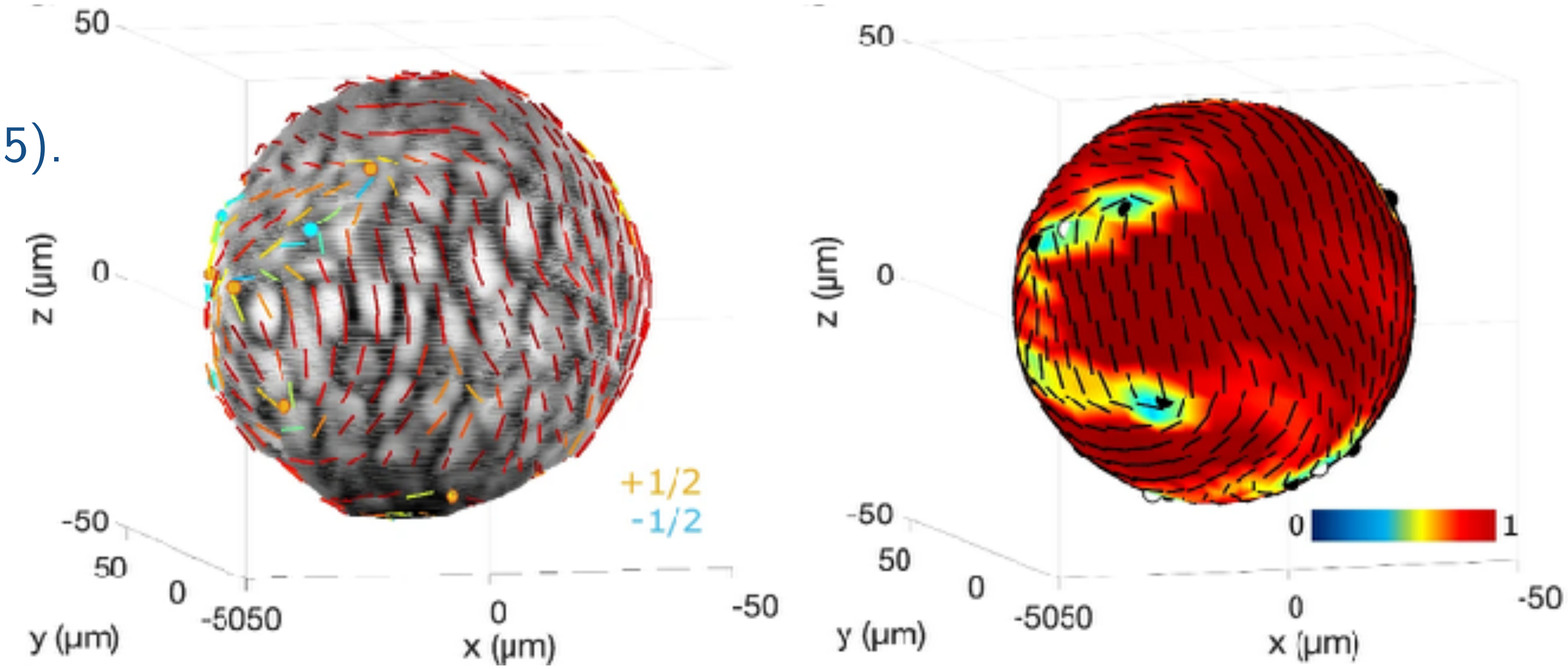


For a general region R with isolated defects at interior points p_i, let D_i be a small disc around each defect. Then

$$\begin{aligned}\int_{\partial R}\mathbf{n}_\perp\cdot d\mathbf{n} &= \int_{\partial(R\setminus\cup_i D_i)}\mathbf{n}_\perp\cdot d\mathbf{n}+\sum_i\int_{\partial D_i}\left(d\theta+\mathbf{e}_2\cdot d\mathbf{e}_1\right),\\ &= \int_{R\setminus\cup_i D_i} d\left(\mathbf{n}_\perp\cdot d\mathbf{n}\right)+\sum_i\left(2\pi w_{p_i}-\int_{\partial D_i}K_{\mathsf{G}}\,dA\right),\\ &= -\int_R K_{\mathsf{G}}\,dA+2\pi\sum_i w_{p_i}\,.\end{aligned}$$

This is an instance of the **Gauss-Bonnet-Chern theorem**.

V. A Digression onto Curved Surfaces

$$\int_{\partial R} \mathbf{n}_\perp \cdot d\mathbf{n} = -\int_R K_\mathsf{G}\, dA + 2\pi \sum_i w_{p_i}\,.$$

This is an instance of the **Gauss-Bonnet-Chern theorem**[†]

$$\int_{\partial R} \omega - \int_R \Omega = 2\pi\, e[R],$$

where e is the **Euler class** of the vector bundle, here the tangent bundle to the curved surface.

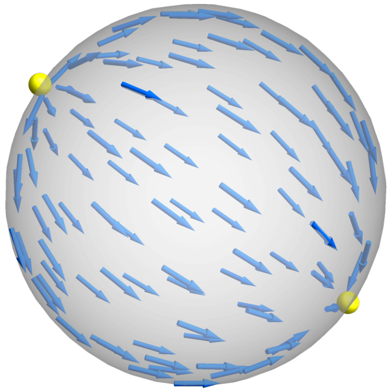
The identification of the Euler number $e[R]$ with the sum of winding numbers of the defects is, in turn, an instance of the **Poincaré-Hopf index theorem**.

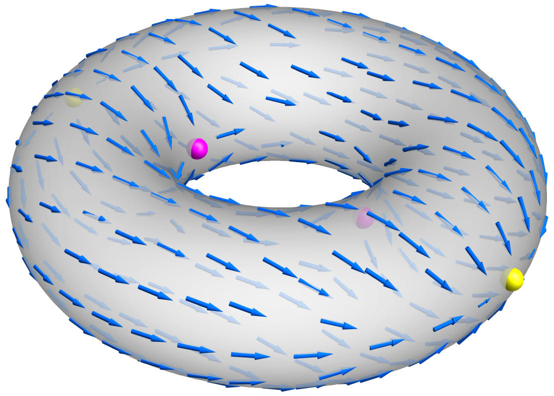
[†] I am taking liberties by not being careful to specify what the boundary conditions are; for correct statements see Milnor & Stasheff (1974) or Bott & Tu (1982).

V. A Digression onto Curved Surfaces

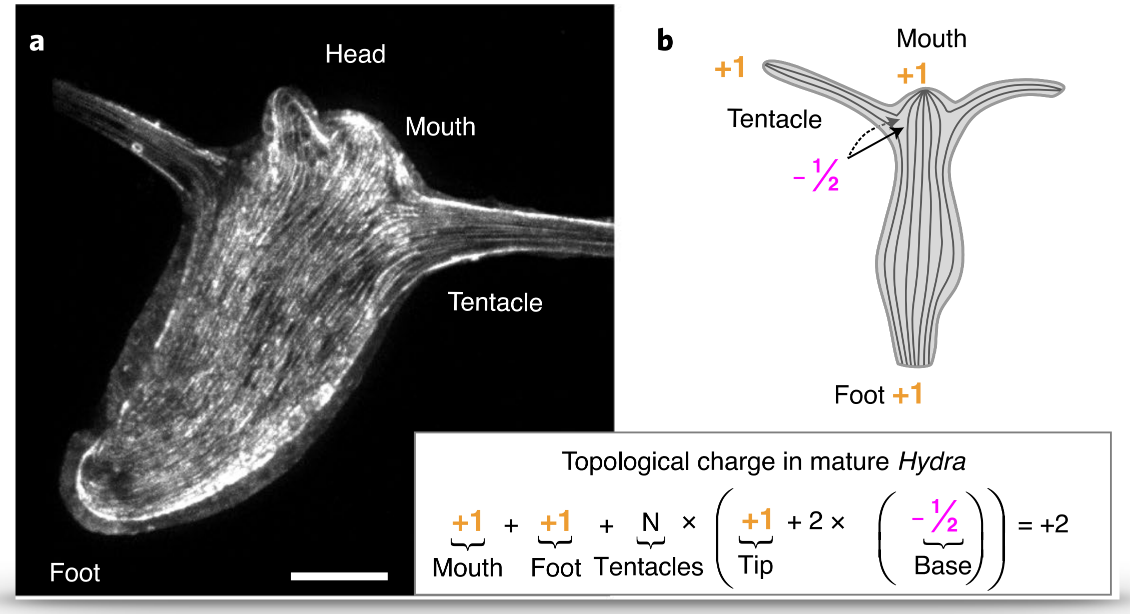
The identification of the Euler number $e[R]$ with the sum of winding numbers of the defects is, in turn, an instance of the **Poincaré-Hopf index theorem**.

For a closed orientable surface, the sum of the winding numbers[†] of all defects in any tangent vector field is a topological invariant, the **Euler characteristic**.





Euler characteristic $\chi = 1 + 1 = 2$ $\chi = 1 - 1 - 1 + 1 = 0$

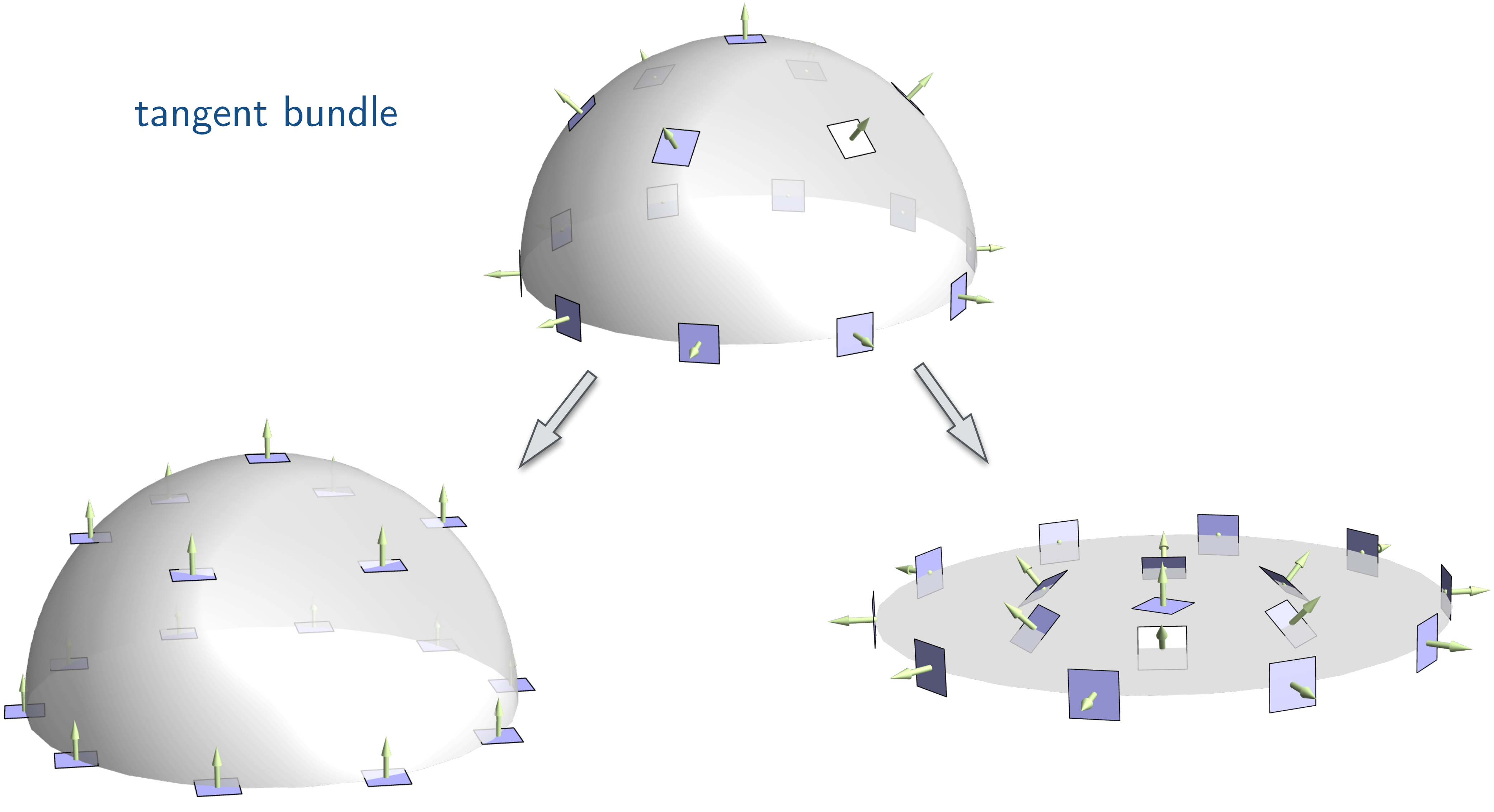


Y. Maroudas-Sacks *et al.*
Nat. Phys. **17**, 251 (2021).

[†] Also called indices, as in the name of the theorem.

V. A Digression onto Curved Surfaces

The curvature of vector fields (textures of orientation) we have introduced here can be split into two types …



tangent bundle

flat vector bundle, curved space

curved vector bundle, flat space

The textures we will talk about next will be curved vector bundles over flat spaces.

Lecture 1: Textures in the Plane

Lecture 2: Escape from the Plane

Lecture 3: Hopfions and Chiral Topology

Lecture 4: Practicals — examples & discussion

I. Textures and their Defects

II. Connection and Winding Numbers

III. Schlieren Textures and Pontrjagin-Thom Construction

IV. On the exterior calculus

V. A Digression onto Curved Surfaces