Lecture 2: [Radu Ignat, Budapest 2025]

Mathematical analysis of domain walls

→ topological solitons in phase transitions models

→ transition layers between two minimal states of a potential function

Analysis: two cases ⇔ finite number of wells
⇔ infinite (continuum of wells) of the potential

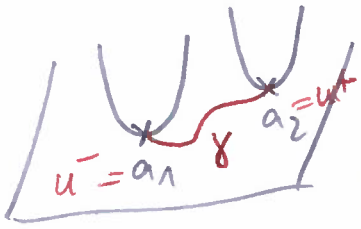
I. Allen-Cahn model

→ phase transitions

$\Omega \subset \mathbb{R}^N$ domain, $u: \Omega \to \mathbb{R}^d$, $\varepsilon > 0$

$$E_\varepsilon(u) = \int_\Omega \varepsilon \frac{|\nabla u|^2}{2} + \frac{1}{\varepsilon} F(u)\, dx$$

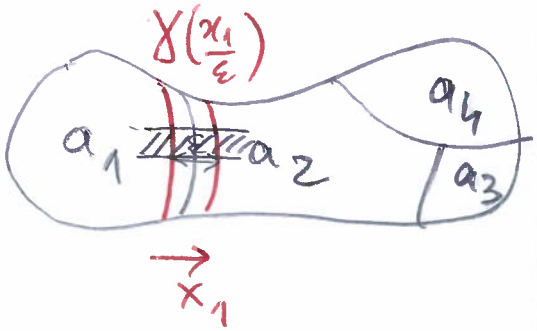
$F: \mathbb{R}^d \to \mathbb{R}$, $F \geq 0$ and $\{F=0\} = \{a_1, \dots, a_K\}$ finite



Ex: $d=1$ (scalar model)

$$F(u) = (1-u^2)^2, \quad \{F=0\} = \{\pm 1\}$$

AIM: Analyse the asymptotics of E_ε as $\varepsilon \to 0$



Expectation: as $\varepsilon \to 0$, limit configurations $u_*: \Omega \to \{F=0\}$

→ nucleation of domain walls

Domain wall: transition layer from a_j to a_k, $j \neq k$

→ study critical points u_ε of E_ε for $\varepsilon > 0$ small

(existence, uniqueness, regularity, properties...)

→ (PDE) $-\Delta u_\varepsilon + \frac{1}{\varepsilon^2} \nabla F(u_\varepsilon) = 0$ in Ω

! system of d PDEs

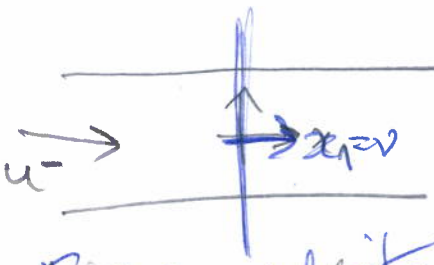
Study in a strip $\Omega = \mathbb{R} \times \mathbb{T}^{N-1}$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ torus

$u : \Omega \to \mathbb{R}^d$ periodic in $x_2, \dots, x_N$

Boundary conditions:

(B.C.) $u \to u^\pm$ as $x_1 \to \pm\infty$

with $F(u^\pm) = 0$.



~~For simplicity~~ ($\varepsilon = 1$):

$$E(u) = \int_\Omega \frac{|\nabla u|^2}{2} + F(u)\, dx$$

~~of notation~~

Minimisation:

$$\min_{(B.C.)} E(u) \qquad (MIM)$$

Expectation: the domain wall is 1D,

$$u = u(x_1) \Rightarrow E(u) = \int_{\mathbb{R}} \frac{|u'|^2}{2} + F(u)\, dx_1$$

ODE: $-u'' + \nabla F(u) = 0$ in $\mathbb{R}$

$\Rightarrow$ equipartition: $\frac{|u'|^2}{2} = F(u)$ in $\mathbb{R}$

$a^2 = b^2$

$a^2 + b^2 = 2|ab|$

$$\Rightarrow E(u) = \int_{\mathbb{R}} \sqrt{2F(u)}\, |u'|\, dx_1 = \text{geod}(u^-, u^+)$$

degenerate metric

in $(\mathbb{R}^d, \sqrt{2F}\, dx^2)$

Thm: Every minimizer in (MIM) is 1D.

Proof:

$$E(u) = \int_{\mathbb{R}} \int_{\mathbb{T}^{N-1}} \frac{1}{2} |\nabla u|^2 + F(u)\, dx_1\, dx', \quad x' = (x_2, \dots, x_N)$$

$|\nabla u|^2 = |\partial_1 u|^2 + |\nabla_{x'} u|^2 \geq 0$

$$\geq \int_{\mathbb{T}^{N-1}} dx' \int_{\mathbb{R}} \frac{1}{2} |\partial_1 u|^2 + F(u)\, dx_1 \geq \text{geod}(u^-, u^+)$$

with $=$ iff $\nabla_{x'} u = 0 \iff u = u(x_1)$. ∎

→ existence : $F(u) > 0$ as $|u| \to \infty$

YES if $\operatorname{grad}(u^-, u^+) < \operatorname{grad}(u^-, a) + \operatorname{grad}(a, u^+)$ $\forall a \in \{F=0\}$

→ uniqueness : No in general (the geodesic is not unique in general : $\{u^\pm\} = \{\text{poles of sphere}\} \Rightarrow$ infinitely many geodesics)

if geodesic (u^-, u^+) is unique, then uniqueness up to translation

→ regularity : $u \in C^\infty$ if F is C^∞

(!) $E_\varepsilon(u_\varepsilon) \to \min$

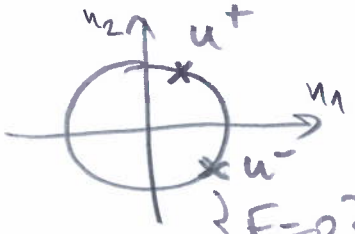
$u_\varepsilon = u^{1D}\left(\frac{x_1}{\varepsilon}\right)$

$\Rightarrow \min E_\varepsilon(u_\varepsilon) = \mathcal{O}(1) = \operatorname{grad}(u^+, u^-)$

II. Ginzburg-Landau type model

Here, $\{F=0\}$ is infinite (continuum of zeros)

Ex : $d=2$, $F(u) = (1-|u|^2)^2$



$\{F=0\} = \mathbb{S}^1$

if $u^\pm \in \mathbb{S}^1$, $u^- \neq u^+$, then

$\inf E(u) = 0$ (B.C.)

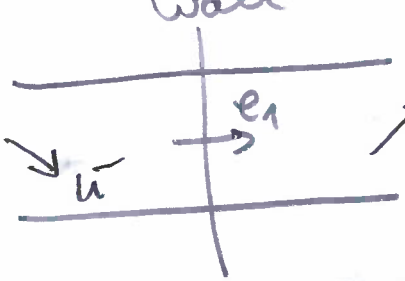
$\Rightarrow$ No minimizers in (MIM)

(MIM-prediction)

NEW setting : $N = d \geq 2$

$u : \mathbb{R} \times \mathbb{T}^{N-1} \to \mathbb{R}^N$, $\boxed{\nabla \cdot u = 0}$

→ motivation : micromagnetics, smectic liquid crystals ($N=2$)



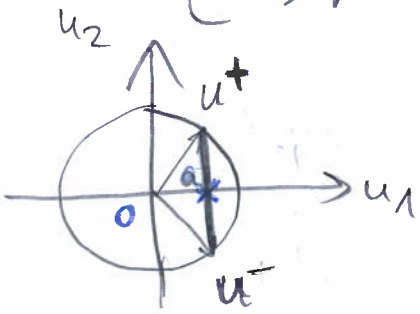
Impose : $u^+ \cdot e_1 = u^- \cdot e_1 = a$

(because Stokes : $0 = \int_\Omega \nabla \cdot u = \int_{\partial\Omega} u \cdot e_1$)

NEW PDE $\quad -\Delta u + \nabla F(u) = \nabla p$, $p = (-\Delta)^{-1}(\nabla^2 F(u) : \nabla u)$

Frobenius

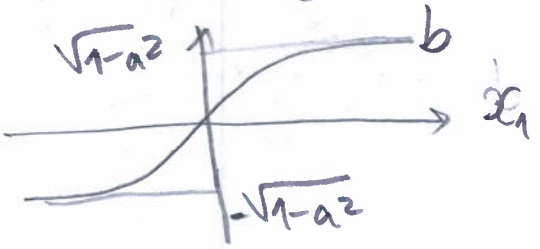
Case 1: $N=d=2$, $F(u)=\frac{1}{2}(1-|u|^2)^2$, $\nabla\cdot u=0$
(→ Aviles-Giga model)



1D walls: $u^{1D}=u^{1D}(x_1)=(a,b(x_1))$
$\nabla\cdot u^{1D}=0$

$$E(u^{1D})=\int_{\mathbb{R}}\frac{1}{2}(b')^2+\frac{1}{2}(1-a^2-b^2)^2\,dx_1$$

equipartition: $(b')^2=(1-a^2-b^2)^2 \Rightarrow b(x_1)=\sqrt{1-a^2}\tanh\left(\sqrt{1-a^2}\,x_1\right)$

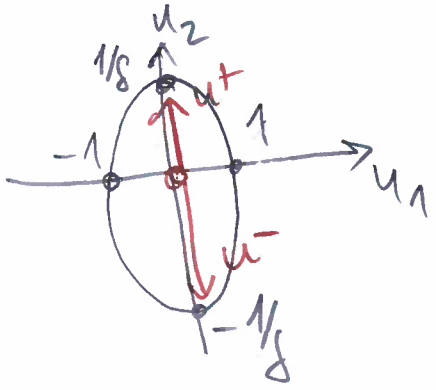


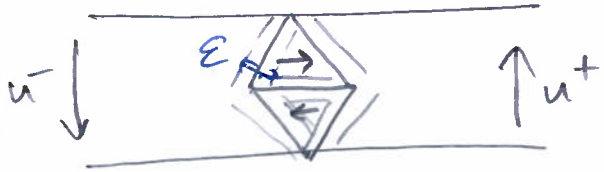
$\Rightarrow E(u^{1D})=\frac{|u^+-u^-|^3}{6}$

Thm (Jin-Kohn '00)
u^{1D} is a min in (MIM).

RK: $F(u)=\frac{1}{\varepsilon^2}\left(1-u_1^2-\delta^2u_2^2\right)^2$ with ε small
$\Rightarrow$ the min in (MIM) is **not** 1D

($\delta<\frac{1}{\sqrt{3}}$) small





$(\varepsilon\to 0)$

microstructure (diamond)

Uniqueness: **Thm (I-Monteil '20)**

if $F(u)=\frac{1}{2}(1-|u|^2)^2$, then u^{1D} is the unique minimizer in $(\pi/4)$ (up to translations).

Proof: Use **calibrations**: $\phi:\mathbb{R}^2\to\mathbb{R}^2$ s.t. (to prove minimality)

for minimality:

1) $\int_\Omega \nabla\cdot[\phi(u)] \le E(u)$, $\forall u:\Omega\to\mathbb{R}^2$, $\nabla\cdot u=0$ + (B.C.)
(lower bound — ~~boundary~~ energy)

2) (Saturation in 1) $\exists u^*$ (for us $u^*=u^{1D}$) s.t. $\int_\Omega \nabla\cdot[\phi(u^*)] = E(u^*)$

Sketch: $F(u)=\frac{1}{2}w(u)^2$, $w(u)=1-|u|^2$

(!) $\Box w(u) = \partial_{11}w(u) - \partial_{22}w(u) = 0$

$\Rightarrow \exists\, \alpha:\mathbb{R}^2\to\mathbb{R}$, $(\alpha=\frac{1}{2}\nabla\cdot\phi)$

$$\nabla\phi(u)=\begin{pmatrix} \alpha(u) & w(u) \\ w(u) & \alpha(u)\end{pmatrix} = (\partial_j\phi_i)$$

$=\nabla\phi_1$ (first row), $=\nabla\phi_2$ (second row)

Then for $\nabla\cdot u=0$,

$$\begin{aligned}\nabla\cdot[\phi(u)] &= \partial_1[\phi_1(u)] + \partial_2[\phi_2(u)] \\ &= \nabla\phi_1(u)\cdot\partial_1 u + \nabla\phi_2(u)\,\partial_2 u \\ &= \alpha(u)(\partial_1u_1 + \partial_2u_2) + w(u)(\partial_1u_2+\partial_2u_1) \\ &= \frac{1}{2}(\partial_1u_2+\partial_2u_1)^2 + \underbrace{\frac{1}{2}w(u)^2}_{=F(u)} - \frac{1}{2}(w(u)-\partial_1u_2-\partial_2u_1)^2 \\ &= \frac{1}{2}|\nabla u|^2 - \underbrace{\frac{1}{2}(\partial_1u_1-\partial_2u_2)^2}_{\ge 0} - \underbrace{(\partial_1u_1\partial_2u_2-\partial_1u_2\partial_2u_1)}_{\nabla^\perp\cdot\begin{pmatrix}u_1\partial_2u_2\\ -u_1\partial_1u_2\end{pmatrix}} \\ &\quad + F(u) - \underbrace{\frac{1}{2}(w(u)-\partial_1u_2-\partial_2u_1)^2}_{\ge 0}\end{aligned}$$

$\Rightarrow$ 1) as $\int_\Omega I = 0$

and 2) as $\partial_1 u_1^{1D} = 0 = \partial_2 u_2^{1D}$, $w(u^{1D}) = (1-|u^{1D}|^2)$

$= \partial_1 u_2^{1D}$, $\partial_2 u_1 = 0$.

equipartition

Uniqueness: if u is min $\Rightarrow$ saturation $\Rightarrow$

$\Rightarrow \partial_1 u_1 = \partial_2 u_2 = 0 \Rightarrow u_2 = u_2(x_1)$, $u_1 = u_1(x_2)$

$(\nabla \cdot u = 0)$

$\Rightarrow$

$u_1 = a$ at $\pm\infty$ $\Rightarrow u_1 \equiv a \Rightarrow u = u(x_1)$. ∎

→ **Link** with scalar conservation laws:

$F(u)=0 \Leftrightarrow |u|=1$, $\nabla \cdot u = 0 \Rightarrow 0 = \partial_1 u_1 + \partial_2 u_2$

$= \pm\sqrt{1-u_1^2}$ $\quad f(u_1)$

$\Rightarrow 0 = \partial_t u_1 + \partial_s[f(u_1)]$ scalar conservation law

⇒ Calibration ϕ ⇝ couple (entropy, entropy-flux)

→ DeSimone-Kohn-Müller-Otto '02, Ambrosio-DeLellis-Mantegazza '01

Jabin-Perthame '02

Generalization:

→ work with Roger Moser

How about higher dimension $N \geq 3$?

$u : \mathbb{R} \times \mathbb{T}^{N-1} \to \mathbb{R}^N$, $\nabla \cdot u = 0$

Ex: $F(u) = \frac{1}{2}(1-|u|^2)^2$, $u^\pm \in S^{N-1} = \{F=0\}$

$u^+ \cdot e_1 = u^- \cdot e_1$



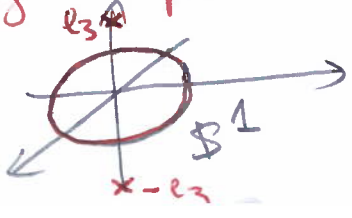
ℓ-period

$\Rightarrow \inf E(u) = 0$ (B.C.)

$(u_n = \gamma(\frac{x_1}{n})$, $F(\gamma)=0)$

Good potential: $(N=3)$ $u=(u_1,u_2,u_3)$

$F(u) = \frac{1}{2}(1-|u|^2)^2 + 2u_3^2(u_1^2+u_2^2)$



Thm: Any min of $E(u)$ under (B.C.) is 1D.

(I-Monteil)

$F(u) = \frac{1}{4} \sum_{i \neq j} |\partial_j \phi_i|^2$

Sketch: $\phi(u) = \nabla \psi(u)$, $\psi(u) = -u_1 u_2 \left(\frac{u_1^2 + u_2^2}{2} + u_3^2 - 1 \right)$,

$\Rightarrow \nabla \phi(u)$ symmetric, $\left[\Pi^+ A = \frac{1}{2}(A + A^t) \text{ projection of sym.} \right]$

$\Pi_0 A = A - \frac{\mathrm{tr}(A)}{N} I$ projection on traceless

$\nabla \cdot [\phi(u)] = \nabla \phi(u) : (\nabla u) \overset{\nabla \cdot u = 0}{=} \nabla \phi(u) : \Pi_0 \nabla u$

$= \Pi_0 \nabla \phi(u) : \Pi^+ \nabla u \qquad [\partial_1 \phi_1 = \dots = \partial_N \phi_N]$

$\leq \frac{1}{2} |\Pi_0 \nabla \phi(u)|^2 + |\Pi^+ \nabla u|^2$

$= F(u)$

(!) $\int_\Omega |\Pi^+ \nabla u|^2 = \frac{1}{2} \int_\Omega |\nabla u|^2$ if $\nabla \cdot u = 0$. ∎

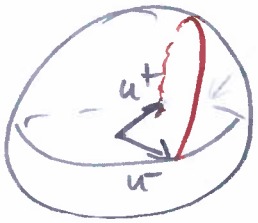
$\Rightarrow$ 1).

Bloch walls: $N = 2$, $\Omega = \mathbb{R} \times \mathbb{T}$, $M = (u, M_3) : \Omega \to \mathbb{S}^2$

$\nabla \cdot u = 0$ (zero stray field)

$E(m) = \int_\Omega (|\nabla m|^2 + M_3^2) \, dx$

limit states: $\{M_3 = 0\} = \mathbb{S}^1$



$u^+ \cdot e_1 = e_1^- \cdot e_1 = a$

$\Rightarrow$ 1D wall: $M^{1D} = (a, \sqrt{1-a^2} \cos \varphi(x_1), \sqrt{1-a^2} \sin \varphi(x_1))$

equipartition: $E(u^{1D}) = 2 \int_{\mathbb{R}} |\dot{u}^{1D}| \, |u_3^{1D}| \, dx_1$

$= |u^+ - u^-|^2$

Thm (I-Murlat '12): if $u^\pm = \{ \pm e_2 \}$ (180° Bloch wall),

then u^{1D} is the unique min of E (up to translations).

Proof: Use calibrations: $\phi: \mathbb{S}^2 \to \mathbb{R}^2$

$$\phi(M) = \mathcal{Z}\big(M_2(1-M_1^2),\, M_1(1-M_2^2)\big)$$

$$\Rightarrow \quad \nabla\cdot[\phi(M)] \le |\nabla M|^2 + M_3^2 + 2\nabla\cdot\begin{bmatrix} M_2\partial_2 M_1 \\ -M_2\partial_1 M_1\end{bmatrix}$$

$\nabla\cdot(M_1,M_2)=0$

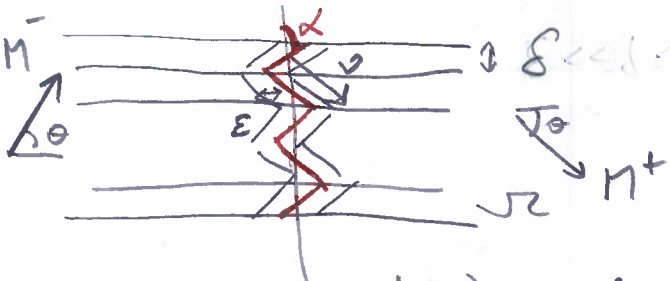
$$\Rightarrow \int \nabla\cdot[\phi(M)] \le E(M)$$

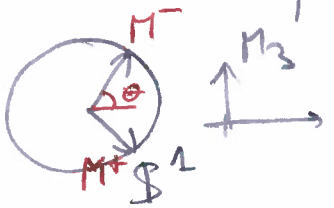
and "=" iff $M = M^{1D}$

Zigzag walls: $M: \Omega \to \mathbb{S}^2$, $\Omega = \mathbb{R}\times\mathbb{T}$, $\nabla\cdot M' = 0$ $(M' = (M_1, M_2))$

$$E_\varepsilon(m) = \int_\Omega \varepsilon|\nabla M|^2 + \frac{1}{\varepsilon}M_2^2$$

limit states: $\{M_2 = 0\}$



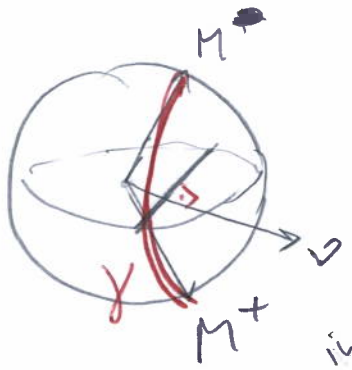


$M^+\cdot e_1 = M^-\cdot e_1 = \cos\theta$

Thm (I-Mosn '12) if $M \to M^\pm$ as $x_1 \to \pm\infty$, m $\theta \le \theta_0$

then $\inf_{(B.C.)} E_\varepsilon(M) = 4F(\theta) + o(1)$, $\varepsilon \to 0$

with $F(\theta) = \sin\theta - \theta\cos\theta \simeq \theta^3$ for $\theta \le \theta_0$.



1D wall (Bloch): $M_\varepsilon = M_\varepsilon(t\nu) = \gamma(\frac{t}{\varepsilon})$

$$\inf_\alpha E_\varepsilon(M_\varepsilon)/\text{length} = \inf_\alpha \left[2\int_{\mathbb{T}^2} |\gamma_2|\,|\gamma'|\,dt \Big/ \cos\alpha\right]$$

$$\Big(!\ \alpha \to \tfrac{\pi}{2}\Big) \quad = 4F(\theta) + o(1)$$

Cross-tie wall : $M : \Omega \to \mathbb{S}^2$



$$E_\varepsilon(M) = \int_\Omega \varepsilon |\nabla M|^2 + \frac{1}{\beta} m_3^2 + \frac{1}{\varepsilon} \int_\Omega \| \nabla^{-1} \partial_1 M' \|^2$$

with $\beta \ll \varepsilon \ll 1$

(M prefers to be in $\mathbb{S}^1$, and then $\nabla \cdot M' = 0$)

→ Alouges - Riviere - Serfaty '02