Mathematical analysis for vortices

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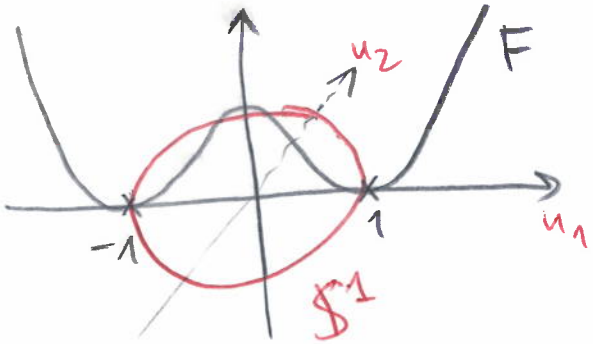
→ a baby model: Ginzburg-Landau (G-L)
(superconductivity, BEC, micromagnetics, liquid crystals...)

2D Ginzburg-Landau Model:

$\Omega \subset \mathbb{R}^2$ domain, $u: \Omega \to \mathbb{R}^2 \simeq \mathbb{C}$, $\varepsilon > 0$
(2D vector field)

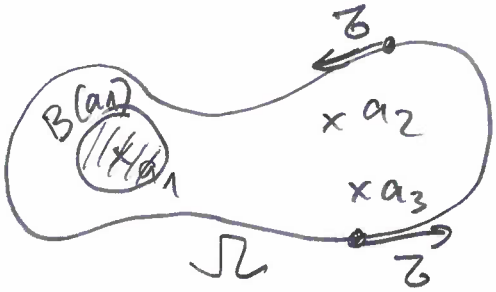
$$E_\varepsilon(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}\underbrace{(1-|u|^2)^2}_{=F(u)}\,dx$$

$\nabla u = (\partial_1 u \ \partial_2 u)$
$|u| = \sqrt{u_1^2+u_2^2}$ euclidian norm



$$\mathbb{S}^1 = \{u \in \mathbb{R}^2 : F(u)=0\}$$
$u = (u_1, u_2)$

As $\varepsilon \to 0$, limit configurations $u_* : \Omega \to \mathbb{S}^1$
→ harmonic map problem (HMP)
→ nucleation of vortices



(up to a rotation)

$$u_* \simeq \left(\frac{x-a_k}{|x-a_k|}\right)^{d_k} \in \mathbb{C}$$ around a_k (for $x \sim a_k$)

a_k = vortex of topo. degree $d_k \in \mathbb{Z}^*$

"infinite" limit energy: $\int_{B(a_k)} |\nabla u_*|^2\,dx = +\infty$.

Goal: Analyse the asymptotics of E_ε as $\varepsilon \to 0$.

Boundary data: $g : \partial\Omega \to \mathbb{S}^1$ smooth

Topological degree (winding number): $\deg(g) = \frac{1}{2\pi} \int_{\partial\Omega} \underbrace{g \times \partial_\tau g}_{= \det[g\ \partial_\tau g]} \, d\tau \in \mathbb{Z}$

Minimisation:

$$\min_{u:\Omega\to\mathbb{R}^2} E_\varepsilon(u) \quad \text{s.t. } u|_{\partial\Omega} = g \qquad \text{(MIN)}$$

* existence, uniqueness, regularity & properties (→ vortices, symmetry...)

Existence: → direct method in Calculus of variations

if u_ε is a min in (MIN), then $\frac{d}{dt}\Big|_{t=0} E_\varepsilon(u_\varepsilon + t\zeta) = 0$

$\forall \zeta$ test function, $\zeta = 0$ $\partial\Omega$

$$\text{(PDE)} \quad \begin{cases} -\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) & \text{in } \Omega \\ u_\varepsilon = g & \partial\Omega \end{cases}$$

(!) System of 2 scalar PDEs

Uniqueness: YES if ε is large (where E_ε is str. convex)

NOT always if ε is small

Regularity: u_ε is smooth $(C^\infty(\bar\Omega))$; $|u_\varepsilon| \le 1$ in Ω

Two cases: $\deg(g) = 0$ / $\deg(g) \neq 0$.

Case 1: $\deg(g)=0$, $g=e^{i\varphi_0}$, $\varphi_0:\partial\Omega\to\mathbb{R}$ smooth

→ uniqueness of u_ε if ε is small (Ye-Zhou '96)

Thm (Bethuel-Brezis-Hélein '93)

! still a gap $\varepsilon\in(\varepsilon_0,\varepsilon_1)$

As $\varepsilon\to 0$, $u_\varepsilon\to u_*$ [in $H^1\cap C^k_{loc}\cap C^{1,\alpha}(\bar\Omega)$, $\forall k$, $\forall\alpha\in(0,1)$]

where $u_*:\Omega\to S^1$ is the unique sol

(HTP) $\begin{cases} -\Delta u_* = |\nabla u_*|^2 u_* & \text{in } \Omega \\ u_* = g & \partial\Omega \end{cases}$

More precisely, $u_*=e^{i\varphi_*}$, $\varphi_*:\Omega\to\mathbb{R}$ is the unique sol. of $\begin{cases}\Delta\varphi_*=0 & \Omega\\ \varphi_*=\varphi_0 & \partial\Omega\end{cases}$. Also, $E_\varepsilon(u_\varepsilon)=\frac12\int_\Omega|\nabla u_*|^2+o(1)$ as $\varepsilon\to 0$.

Rk: $|u_\varepsilon|\to|u_*|=1$ unif in Ω ⇒ u_ε has **no** zeros

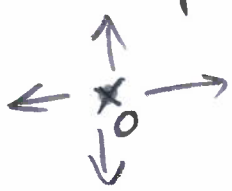
⇒ No VORTICES

Case 2: $\deg g=d\neq 0$ (assume $d>0$)

$g=e^{i\varphi_0}$, φ_0 has jumps

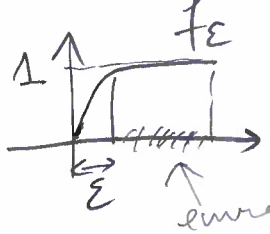
We expect vortices:

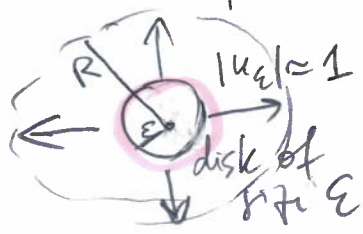
1) $\varepsilon\to 0$: prototype of vortex degree +1



$u_*=\frac{x}{|x|}=e^{i\theta}$! 0 is a singularity

2) $\varepsilon>0$: prototype $u_\varepsilon(x)=f_\varepsilon(|x|)\frac{x}{|x|}$





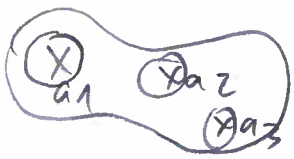
0 is a topological vortex of u_ε of degree 1.

(4) NEW

$$E_\varepsilon(u_\varepsilon) \simeq \int_{B_R \setminus B_\varepsilon} \frac{1}{2} \underbrace{\left|\nabla\left(\frac{x}{|x|}\right)\right|^2}_{=\frac{1}{|x|^2}} = \pi \int_\varepsilon^R \frac{dr}{r} = \boxed{\pi |\log \varepsilon| + O(1)} \to \infty, \ \varepsilon \to 0$$

Thm (BBH '94) if $\deg(g) = d > 0$, then for a subsequence

$$u_\varepsilon \to u_* = e^{i\varphi_*} \frac{x-a_1}{|x-a_1|} \cdots \frac{x-a_d}{|x-a_d|} \quad \left[\text{in } C^k_{loc}(\Omega \setminus \{a_\ell\}_\ell) \cap C^{1,\alpha}_{loc}(\overline{\Omega} \setminus \{a_\ell\}_\ell)\right]$$



where $\Delta \varphi_* = 0$ in Ω, $u_* = g$ on $\partial\Omega$

and $a_1, \dots, a_d$ are distinct vortex points of degree ± 1. Moreover, $E_\varepsilon(u_\varepsilon) = \pi d |\log \varepsilon| + \underline{\underline{O(1)}}$.

Renormalized energy: $O(1) = \underbrace{W(a_1, \dots, a_d)}_{\text{R.E.}} + \underbrace{d\gamma}_{\substack{\text{energy of} \\ \text{a radial} \\ \text{profile}}} + o(1)$



where

$$W(a_1, \dots, a_d) = \lim_{r \to 0} \left(\int_{\Omega \setminus \bigcup_1^d B(a_j, r)} \frac{1}{2} |\nabla u_*|^2 dx - \pi d |\log r| \right)$$

$$\simeq -\pi \sum_{j \neq \ell} \log |a_j - a_\ell| + \dots$$

$\Rightarrow$ repulsion of vortices

$$\gamma = \lim_{\varepsilon \downarrow 0} \left(E_\varepsilon(u_\varepsilon, B_1) - \pi |\log \varepsilon| \right) \text{ for the data } u_\varepsilon = x \text{ on } \partial B_1$$

(!) $(a_1, \dots, a_d)$ is a min of W.

Jacobian → detects the vortices

$$jac(u_\varepsilon) = det(\nabla u_\varepsilon) = \partial_1 u_\varepsilon \times \partial_2 u_\varepsilon$$

$$= \frac{1}{2}\left[\partial_1(u_\varepsilon \times \partial_2 u_\varepsilon) - \partial_2(u_\varepsilon \times \partial_1 u_\varepsilon)\right]$$

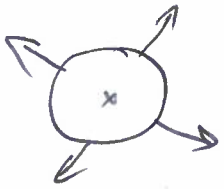
$$= \frac{1}{2} \nabla \times \left[u_\varepsilon \times \nabla u_\varepsilon\right]$$

(Jerrard-Sour '02)

$$\left(E_\varepsilon(u_\varepsilon) \geq \|jac(u_\varepsilon)\| |\log\varepsilon| + \mathcal{M} + o(1)\right)$$

$$\xrightarrow[\varepsilon \to 0]{} jac(u_*) = \pi \sum_1^d \delta_{a_k}$$

Uniqueness of minimizers:



* YES if $g(x) = x$ on $\partial\Omega = \partial B_1$

Pacard-Riviere '00 : if $\varepsilon > 0$ small, then

$u_\varepsilon = f_\varepsilon(|x|)\frac{x}{|x|}$ is the unique min of E_ε.

$\left(E_\varepsilon(u_\varepsilon) = \pi|\log\varepsilon| + \gamma + o(1),\ \varepsilon \to 0\right)$

* NO if $g(e^{i\theta}) = e^{2i\theta}$ on ∂B_1.



u_ε is not unique: group action

$$\alpha \in \mathbb{R} \longmapsto e^{-2i\alpha} u(e^{i\alpha}x)$$

keeps invariant g and E_ε.

→ the same theory is valid for micromagnetic vortices

Hang-Lin '01