Polytopo training school – 12/13 May 2025 – Budapest

Lecture notes of M. Garst:

Theory of magnetism and magnetic materials – topological textures

In quantum magnetism interacting many-spin systems are studied which are described eg. by the Heisenberg model. Its solution is only known in special cases and often we rely on approximations or asymptotic expansions. An important asymptotic expansion, that is at the focus of this lecture, uses S – the size of the spin – as a large parameter. In the limit of large S a quasi classical field theory emerges. The field is a unit vector field $\hat{n}(\vec{r},t)$ that represents the local spin orientation. This field theory allows for topologically non-trivial textures.

Spin operators

Spin operators $\vec{S} = \begin{pmatrix} S^x \\ S^y \\ S^z \end{pmatrix}$ obey the algebra of angular momentum

$$\boxed{[S^\alpha, S^\beta] = i\hbar \varepsilon_{\alpha\beta\gamma} S^\gamma \quad \text{and} \quad \vec{S}^2 = \hbar^2 S(S+1)}$$

for a spin $S = 1/2, 1, 3/2, 2, \ldots$ where $\alpha, \beta, \gamma \in \{x, y, z\}$. Alternatively

$$[S^+, S^-] = 2\hbar S^z \quad \text{and} \quad [S^z, S^\pm] = \pm \hbar S^\pm$$

with $S^\pm = S^x \pm i S^y$.

A complete basis is given by the eigenstates of S^z

$$S^z |m\rangle = \hbar m |m\rangle$$

where the quantum number $m \in \{-S, -S+1, \ldots, S-1, S\}$

For example for spin

$$S = 1/2 \quad : \quad m \in \{-1/2, 1/2\}$$
$$S = 1 \quad : \quad m \in \{-1, 0, 1\}$$
$$S = 3/2 \quad : \quad m \in \{-3/2, -1/2, 1/2, 3/2\}$$

The orthonormalized basis $\langle m|m'\rangle = \delta_{m,m'}$ can be constructed with

$$S^{\pm}|m\rangle = \hbar\sqrt{S(S+1) - m(m\pm 1)}\,|m\pm 1\rangle$$

Spin coherent states

The components of the spin operator $\vec{S}$ are generator of the SU(2) group of transformations. The elements of SU(2) can be parametrized by three Euler angles ϕ, θ and χ :

$$R(\phi,\theta,\chi) = e^{i\phi S^z/\hbar}\, e^{i\theta S^y/\hbar}\, e^{i\chi S^z/\hbar}$$

This rotation operator perform three subsequent rotations around the local z-axis, y-axis and z-axis with angles χ, θ and ϕ, respectively.

Applying this operator to the fully polarized spin state $|S\rangle$ with $m = S$ a family of spin states are created

$$|\hat{n}\rangle = R(\phi,\theta,\chi)\,|S\rangle$$

spin coherent states

They are parametrized by the unit vector

$$\hat{n}^T = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta).$$

and they obey

$$(\hat{n}\vec{S})\,|\hat{n}\rangle = \hbar S\,|\hat{n}\rangle \quad \text{and} \quad \langle\hat{n}|\vec{S}|\hat{n}\rangle = \hbar S\hat{n}.$$

As $e^{iXS^z/\hbar}\,|S\rangle = e^{iXS}\,|S\rangle$ the angle X only enters via a phase that we can choose arbitrarily. This corresponds to a gauge freedom that can be eliminated by fixing X. The two independent angles, the polar angle θ and the azimuthal angle ϕ, are defined within the domains $\{\theta\in[0,\pi],\ \phi\in[-\pi,\pi)\}$. The states $|\hat{n}\rangle$ can be expanded in the complete set of eigenstates $\{|m\rangle\}$ and one can show that

$$|\hat{n}\rangle = e^{iSX}\sum_{m=-S}^{S}\frac{\sqrt{2S!}}{\sqrt{(S+m)!\,(S-m)!}}\,e^{im\phi}\left(\cos\frac{\theta}{2}\right)^{S+m}\left(\sin\frac{\theta}{2}\right)^{S-m}|m\rangle$$

The overlap between two coherent states is

$$\langle\hat{n}|\hat{n}'\rangle = \left(\frac{1+\hat{n}\hat{n}'}{2}\right)^{S}e^{-iS\Psi}$$

$$\Psi = 2\arctan\left(\tan\frac{\phi-\phi'}{2}\,\frac{\cos\frac{\theta+\theta'}{2}}{\cos\frac{\theta-\theta'}{2}}\right)+\chi-\chi'$$

ie. they are not orthogonal. Moreover, they form an overcomplete basis as

$$\mathbb{1} = \int d\mu(\hat{n})\,|\hat{n}\rangle\langle\hat{n}|$$

where

$$d\mu(\hat{n}) = \frac{2S+1}{4\pi}\,d\hat{n}$$

with $d\hat{n} = \sin\theta\,d\theta\,d\phi$ is the Haar measure of the $SU(2)$ group. One can also show that

$$\hbar(S+1)\int d\mu(\hat{n})\,\hat{n}_\alpha\,|\hat{n}\rangle\langle\hat{n}| = \hat{S}^\alpha.$$

For the proofs of these relations, see eg. A. Auerbach, Interacting Electrons and Quantum Magnetism.

The classical limit is achieved by sending $S \to \infty$. In this limit, the overlap of different coherent states vanishes exponentially with S:

$$|\langle \hat{n}|\hat{n}'\rangle| = \exp\Big[-S \underbrace{\ln \frac{2}{1+\hat{n}\hat{n}'}}_{>0 \text{ for } \hat{n}\neq\hat{n}'}\Big]$$

When the overlap vanishes expectation values of spin operators are functions of unit vectors like classical spins. Quantum effects are therefore associated with the non-orthogonality of coherent states.

The generalization to many spins is straightforward. The many spin coherent state for N spin-S is product of single spin states

$$|\hat{n}\rangle = \prod_{i=1}^{N} |\hat{n}_i\rangle .$$

The overlap and the completeness relation are

$$\langle \hat{n}|\hat{n}'\rangle = \prod_{j=1}^{N}\left[\left(\frac{1+\hat{n}_j\hat{n}_j'}{2}\right)^S e^{-iS\Psi(\hat{n}_j,\hat{n}_j')}\right]$$

$$\int \prod_{j=1}^{N} d\mu(\hat{n}_j)\, |\hat{n}\rangle\langle\hat{n}| = \mathbb{1}$$

Spin path integral

The path integral formalism provides an alternative description of a quantum system that differs from the Hamilton formalism. This formalism is very useful for the treatment of many-particle systems as it allows to formulate mean-field theories and to generate asymptotic expansions. In addition, it is directly related to the notion of quantum field theories. Path integrals are derived with the help of a coherent state representation and, therefore, often allow to consider semiclassical approximations.

We follow standard derivations and consider the partition function of a Hamiltonian H describing a many spin system, $Z = \mathrm{tr}\{e^{-\beta H}\}$, where $\beta = \frac{1}{k_B T}$. It reads in the coherent state representation

$$Z = \int \prod_{j=1}^{N} d\mu(\hat{n}_j)\, \langle \hat{n}|e^{-\beta H}|\hat{n}\rangle$$

It is now convenient to perform a Trotter decomposition and divide β into M so-called imaginary time slices of size $\delta\tau$ such that $\beta = M\delta\tau$:

$$Z = \int \prod_{j=1}^{N} d\mu(\hat{n}_j) \, \langle \hat{n} | \underbrace{e^{-\delta\tau H} \, e^{-\delta\tau H} \, \dots \, e^{-\delta\tau H}}_{M\text{-times with } M\delta\tau = \beta} | \hat{n} \rangle$$

Between each of the factors $e^{-\delta\tau H}$ we now insert a completeness relation of coherent states. In order to distinguish coherent states inserted between different slices we label them $\hat{n}(\tau_n)$ by the imaginary time $\tau_n = n\delta\tau$ with the periodic boundary condition $\hat{n}(0) = \hat{n}(\beta)$:

$$Z = \int \prod_{n=0}^{M-1} \prod_{j=1}^{N} d\mu(\hat{n}_j(\tau_n)) \, \langle \hat{n}(\beta) | e^{-\delta\tau H} | \hat{n}(\tau_{M-1}) \rangle \langle \hat{n}(\tau_{M-1}) | e^{-\delta\tau H} | \hat{n}(\tau_{M-2}) \rangle \dots \langle \hat{n}(\tau_1) | e^{-\delta\tau H} | \hat{n}(0) \rangle$$

Consider one of M terms separately and assume that $\delta\tau \to 0$

$$\langle \hat{n}(\tau_{n+1}) | e^{-\delta\tau H} | \hat{n}(\tau_n) \rangle \approx \langle \hat{n}(\tau_{n+1}) | (1 - \delta\tau H) | \hat{n}(\tau_n) \rangle = \langle \hat{n}(\tau_{n+1}) | \hat{n}(\tau_n) \rangle \left[1 - \delta\tau \langle H \rangle_{\tau_n} \right]$$

In the limit $\delta\tau \to 0$ we can approximate

$$\langle H \rangle_{\tau_n} = \frac{\langle n(\tau_{n+1}) | H | \hat{n}(\tau_n) \rangle}{\langle n(\tau_{n+1}) | \hat{n}(\tau_n) \rangle} \cong \langle \hat{n}(\tau_n) | H | \hat{n}(\tau_n) \rangle = H(\{\hat{n}_j(\tau_n)\}) = H(\hat{n}(\tau_n))$$

In order to obtain this expectation value all spin operators in the Hamiltonian $H(\{\vec{S}_j\})$ can essentially be replaced by the unit vectors $\hat{n}_j$. For the overlap of coherent states we obtain

$$\begin{aligned} \langle \hat{n}(\tau_{n+1}) | \hat{n}(\tau_n) \rangle &= \prod_{j=1}^{N} \left[\left(\frac{1 + \hat{n}(\tau_{n+1}) \hat{n}(\tau_n)}{2} \right)^{s} e^{-is\Psi(\hat{n}_j(\tau_{n+1}), \hat{n}_j(\tau_n))} \right] \\ &\underset{\delta\tau \to 0}{\approx} \prod_{j=1}^{N} \exp\left(-is\delta\tau \left[\frac{\partial}{\partial \tau_{n+1}} \Psi(\hat{n}_j(\tau_{n+1}), \hat{n}_j(\tau_n)) \right]_{\delta\tau = 0} \right) \end{aligned}$$

In the last equation we performed a Taylor expansion assuming boldly that $\hat{n}_j(\tau)$ is a differentiable function.

For the derivative of the phase one obtains

$$\frac{\partial}{\partial\tau_{n+1}}\Psi(\hat{n}_j(\tau_{n+1}),\hat{n}_j(\tau_n))\Big|_{\delta\tau=0} = \frac{\partial}{\partial\tau_{n+1}}\left(2\arctan\left(\tan\frac{\phi_j(\tau_{n+1})-\phi_j(\tau_n)}{2}\,\frac{\cos\frac{\theta_j(\tau_{n+1})+\theta_j(\tau_n)}{2}}{\cos\frac{\theta_j(\tau_{n+1})-\theta_j(\tau_n)}{2}}\right)+\chi_j(\tau_{n+1})-\chi_j(\tau_n)\right)\Big|_{\delta\tau=0}$$

$$= \frac{\partial}{\partial\tau_{n+1}}\Big(\left(\phi_j(\tau_{n+1})-\phi_j(\tau_n)\right)\cos\theta_j(\tau_n)+\chi_j(\tau_{n+1})-\chi_j(\tau_n)\Big)\Big|_{\delta\tau=0}$$

$$= \dot{\phi}_j(\tau_n)\cos\theta_j(\tau_n)+\dot{\chi}_j(\tau_n)$$

Taken together we have for $\delta\tau\to 0$

$$\langle\hat{n}(\tau_{n+1})|e^{-\delta\tau H}|\hat{n}(\tau_n)\rangle \underset{\delta\tau\to 0}{\approx} \exp\left[-iS\delta\tau\sum_{j=1}^{N}\left(\dot{\phi}(\tau_n)\cos\theta(\tau_n)+\dot{\chi}(\tau_n)\right)\right]\exp\left[-\delta\tau\,H(\hat{n}(\tau_n))\right]$$

In the limit $\delta\tau\to 0$ with $N\to\infty$ so that the product $N\delta\tau=\beta$ remains fixed, the partition function reads

$$\begin{aligned}Z &= \lim_{\substack{N\to\infty\\ N\delta\tau=\beta}}\int\prod_{n=0}^{N-1}\prod_{j=1}^{N}d\mu(\hat{n}_j(\tau_n))\,\exp\left[-\delta\tau\sum_{n=0}^{N-1}\left\{iS\sum_{j=1}^{N}\left(\dot{\phi}_j(\tau_n)\cos\theta_j(\tau_n)+\dot{\chi}_j(\tau_n)\right)+H(\hat{n}(\tau_n))\right\}\right]\\ &= \int_{\hat{n}(0)=\hat{n}(\beta)}\mathcal{D}\hat{n}(\tau)\,\exp\left[+iS\sum_{j=1}^{N}w(\hat{n}_j)-\int_0^\beta d\tau\,H(\hat{n}(\tau))\right]\end{aligned}$$

The limit $N\to\infty$ transforms the product of integrals into a path integral. The symbol $\mathcal{D}\hat{n}$ represents the measure of the path integral that needs to be evaluated with periodic boundary conditions $\hat{n}(0)=\hat{n}(\beta)$. The integral over $d\tau$ in the exponent as well as derivatives ∂_τ represent the discrete definitions in the limit $\delta\tau\to 0$.

We also defined

$$w(\hat{n}) = -\int_0^\beta d\tau\left(\dot{\phi}(\tau)\cos\theta(\tau)+\dot{\chi}(\tau)\right)$$

which gives the Berry phase of the spin history.

Spin Berry phase

Consider the Berry phase for the particular gauge $\dot{\chi} = -\dot{\phi}$:

$$w(\hat{n}) = \int_0^\beta d\tau\ \dot{\phi}\,(1-\cos\theta) = \int_{\phi(0)}^{\phi(\beta)} d\phi\ (1-\cos\theta)$$

This phase is purely geometric as it depends only on the trajectory of $\hat{n}(\tau)$ and not on its explicit time dependence. Consider a path increment $d\hat{n}$ that is connected to the north pole by the longitudes ϕ and $\phi+d\phi$. They enclose the area

$$dw = \int_0^\theta d\theta'\,\sin\theta'\,d\phi = (1-\cos\theta)\,d\phi$$

where $\theta = \theta(\phi)$. Integrating along a closed trajectory with $\hat{n}(0) = \hat{n}(\beta)$ yields the Berry phase w. It measures therefore the area enclosed by $\hat{n}(\tau)$ on the unit sphere. Note that w can be positive or negative depending on the orientation of the trajectory $\hat{n}(\tau)$. At the same time, the Berry phase is only defined up to factors of 4π. The orange area w' enclosed by the trajectory is up to a sign given by

$$w' = \oint d\phi \int_\pi^{\theta(\phi)} d\theta'\,\sin\theta' = \oint d\phi\,(-1-\cos\theta)$$

and corresponds to the gauge choice $\dot{\chi} = \dot{\phi}$. Moreover, $w - w' = \oint d\phi\ 2 = 4\pi$. Additive factors of 4π do not matter because they appear in the exponent and $e^{iS4\pi} = 1$ for all $S = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$.

It is elegant to express the Berry phase in a gauge invariant form

$$w(\hat{n}) = \int_0^\beta d\tau\, \vec{A}(\hat{n})\, \partial_\tau \hat{n}$$

with the vector potential $\vec{A}(\hat{n})$. The choice $\vec{A} = \frac{1-\cos\theta}{\sin\theta}\hat{\phi}$ corresponds to the gauge $\chi = -\phi$ where $\hat{\phi} = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}$. Using Stokes theorem we can rewrite

$$w = \oint d\hat{n}\, \vec{A} = \underbrace{\int dw\, \hat{n}\left(\frac{\partial}{\partial \hat{n}} \times \vec{A}\right)}_{\text{integral over the covered area}}$$

This equation is fulfilled if

$$\frac{\partial}{\partial \hat{n}} \times \vec{A}(\hat{n}) = \hat{n}$$

This corresponds to a magnetic flux in $\hat{n}$-space that has the form of a magnetic monopole. It was shown by P. Dirac that the vector potential of a magnetic monopole is necessarily associated with a Dirac string singularity. For example, for the above choice $\vec{A} = \frac{1-\cos\theta}{\sin\theta}\hat{\phi}$ the vector potential is singular for $\theta = \pi$, ie. at the south pole.

Semiclassical approximation: Landau-Lifshitz equation of motion

The imaginary time path integral expresses the partition function $Z = \int \mathcal{D}\hat{n}\, e^{-S[\hat{n}]}$ in terms of the so-called Euclidian action

$$S[\hat{n}] = -iS \sum_{j=1}^{N} \int_0^\beta d\tau\, \vec{A}(\hat{n}_j)\partial_\tau \hat{n}_j + \int_0^\beta d\tau\, H(\hat{n}(\tau)).$$

The path-integral formalism can also be applied to the evolution operator $U(T) = e^{-iHT/\hbar}$. It is expressed $U = \int \mathcal{D}\hat{n}\, e^{\frac{i}{\hbar}S[\hat{n}]}$ in terms of the real-time action that follows from

the Euclidian action with the help of a Wick rotation from imaginary time $\tau = \frac{it}{\hbar}$ to real time t using $d\tau = \frac{i}{\hbar} dt$ and $\partial_\tau = \frac{\hbar}{i} \partial_t$. It is given by $S[\hat{n}] = \int_0^\tau dt\, L[\hat{n}]$ with the Lagrangian

$$\boxed{L = \hbar S \sum_{j=1}^{N} \vec{A}(\hat{n}_j)\, \partial_t \hat{n}_j - H(\hat{n})}$$

The semiclassical approximation is obtained by considering the saddle point of the action $\frac{\delta S}{\delta \hat{n}} \overset{!}{=} 0$, that yields the Euler-Lagrange equations. For its derivation we implement the constraint $\hat{n}_j^2 = 1$ using Lagrange multipliers λ_j,

$$L = \hbar S \sum_{j=1}^{N} \vec{A}(\hat{n}_j)\, \partial_t \hat{n}_j - H(\hat{n}) - \sum_{j=1}^{N} \lambda_j \left(\hat{n}_j^2 - 1\right).$$

We then obtain for the Euler-Lagrange equation (with $\hat{n}_{i\alpha}$ being the α^{th} component of the i-th spin field)

$$(1) \quad \frac{\partial L}{\partial \lambda_i} = \hat{n}_i^2 - 1 = 0$$

$$(2) \quad \frac{\partial L}{\partial \hat{n}_{i\alpha}} - \partial_t \frac{\partial L}{\partial \partial_t \hat{n}_{i\alpha}} = \hbar S \frac{\partial \vec{A}_\beta(\hat{n}_i)}{\partial \hat{n}_{i\alpha}} \partial_t \hat{n}_{i\beta} - \hbar S\, \partial_t \vec{A}_\alpha(\hat{n}_i) - \frac{\partial H}{\partial \hat{n}_{i\alpha}} - 2\lambda_i \hat{n}_{i\alpha}$$

$$= \hbar S \left(\frac{\partial \vec{A}_\beta(\hat{n}_i)}{\partial \hat{n}_{i\alpha}} - \frac{\partial \vec{A}_\alpha(\hat{n}_i)}{\partial \hat{n}_{i\beta}} \right) \partial_t \hat{n}_{i\beta} - \frac{\partial H}{\partial \hat{n}_{i\alpha}} - 2\lambda_i \hat{n}_{i\alpha} = 0$$

The derivative of the spin gauge field can be simplified using

$$\frac{\partial A_\beta}{\partial \hat{n}_{i\alpha}} - \frac{\partial A_\alpha}{\partial \hat{n}_{i\beta}} = \varepsilon_{\alpha\beta\gamma}\, \hat{n}_\gamma \quad \text{because multiplying with } \varepsilon_{\alpha\beta\delta} \text{ yields}$$

$$\varepsilon_{\alpha\beta\delta} \left(\frac{\partial A_\beta}{\partial \hat{n}_{i\alpha}} - \frac{\partial A_\alpha}{\partial \hat{n}_{i\beta}} \right) = 2 \underbrace{\varepsilon_{\alpha\beta\delta} \frac{\partial A_\beta}{\partial \hat{n}_{i\alpha}}}_{= \hat{n}_{i\delta} \text{ by definition}} = \underbrace{\varepsilon_{\alpha\beta\delta} \varepsilon_{\alpha\beta\gamma}}_{= 2\delta_{\delta\gamma}} n_\gamma$$

$$\Rightarrow \quad (2) \quad \hbar S \left(\partial_t \hat{n}_i \times \hat{n}_i \right)_\alpha - \frac{\partial H}{\partial \hat{n}_{i\alpha}} - 2\lambda_i \hat{n}_{i\alpha} = 0$$

Multiplying (2) with $\hat{n}_{i\alpha}$ and summing over α yields using (1) $\hat{n}_i^2 = 1$:

$$\underbrace{\hbar S\, \hat{n}_i \left(\partial_t \hat{n}_i \times \hat{n}_i\right)}_{=0} - \hat{n}_i \frac{\partial H}{\partial \hat{n}_i} - 2\lambda_i = 0 \quad \rightarrow \quad 2\lambda_i = -\hat{n}_i \frac{\partial H}{\partial \hat{n}_i}$$

Eliminating the Lagrange multiplier λ_i from (2) gives

$$\hbar S \left(\partial_t \hat{n}_i \times \hat{n}_i\right)_\alpha - \left(\delta_{\alpha\beta} - \hat{n}_{i\alpha}\hat{n}_{i\beta}\right) \frac{\partial H}{\partial \hat{n}_{i\beta}} = 0$$

Finally, multiplying with $(\hat{n}_i \times)$ using $\hat{n}_i \times \left(\hat{n}_i \times \partial_t \hat{n}_i\right) = \underbrace{\left(\hat{n}_i \cdot \partial_t \hat{n}_i\right)}_{=0} \hat{n}_i - \underbrace{\left(\hat{n}_i^2\right)}_{=1} \partial_t \hat{n}_i = -\partial_t \hat{n}_i$

as $\partial_t \hat{n}_i^2 = 2 n_i \partial_t \hat{n}_i = 0$

we obtain

$$\boxed{\partial_t \hat{n}_i = \frac{1}{\hbar S}\, \hat{n}_i \times \frac{\partial H}{\partial \hat{n}_i}}$$

The Euler-Lagrange equation yield the Landau-Lifshitz equation describing the precession of the unit vector in an effective magnetic field given by $\frac{\partial H}{\partial \hat{n}_i}$.

The semiclassical equation of motion is controlled in the limit of large S where the saddle point approximation should hold. This can be seen by rescaling time $t \rightarrow S t$ such that $L \rightarrow \hbar \sum_i \vec{A}(\hat{n}_i)\, \partial_t \hat{n}_i - H$ and $\frac{i}{\hbar} S[\hat{n}] \rightarrow \frac{i}{\hbar} S\, S[\hat{n}]$. The large prefactor for $S \rightarrow \infty$ in front of the rescaled action $S[\hat{n}]$ controls the saddle point approximation.

Topological magnetic configurations

The theory $S[\hat{n}]$ depends on the configurations $\hat{n}_i$ for the various spin degrees of freedom $i = 1, 2, \dots, N$. If the spins are regularly distributed on some lattice characterized by some lattice constant a, we can label them instead with their position $\vec{r}_i$; ie $\hat{n}(\vec{r}_i)$. If we are only interested in the low-energy, large lengthscale behavior of the system we can perform the continuum limit $\hat{n}(\vec{r}_i) \to \hat{n}(\vec{r})$ where $\hat{n}(\vec{r})$ represents a mapping from d-dimensional space $\mathbb{R}^d$ to the unit sphere S^2:

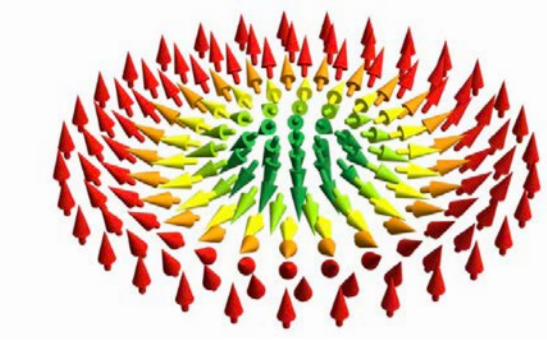
$$\hat{n} : \mathbb{R}^d \to S^2, \quad \vec{r} \mapsto \hat{n}$$

We can "compactify" d-dimensional space by demanding that $\hat{n}(\vec{r}) \to \hat{n}_0$ for $|\vec{r}| \to \infty$. In this case, the map is from a d-sphere to a 2-sphere

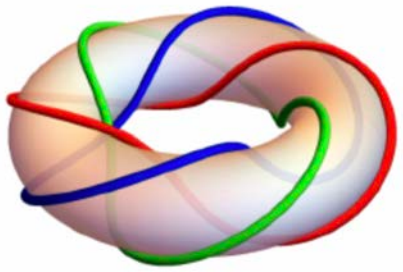
$$\hat{n} : S^d \to S^2, \quad \vec{r} \mapsto \hat{n}$$

Such a map can be topologically non-trivial. Mathematically this is reflected in the corresponding homotopy groups $\pi_d(S^2)$. If $\pi_d(S^2)$ is non-trivial there exist spin textures in d-spatial dimensions.

$d=1$	$\pi_1(S^2) = 0$	no textures in $d=1$
$d=2$	$\pi_2(S^2) = \mathbb{Z}$	(baby-)skyrmion textures in $d=2$
$d=3$	$\pi_3(S^2) = \mathbb{Z}$	Hopfion textures in $d=3$.



Skyrmion



Hopfion

As $\pi_1(S^2) = 0$ there exist no textures in $d = 1$. However if the unit vector $\hat{n}$ is forced into a plane eg. by strong anisotropies, it does not cover a 2-sphere S^2 but only a circle S^1. In this case, there exist also textures in $d = 1$ as the mapping from a circle to a circle is non-trivial

$d = 1 \qquad \pi_1(S^1) = \mathbb{Z}$ domain wall textures in $d = 1$.



2π - domain wall